

Homotopy field theory in dimension 2 and group-algebras

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Abstract

We apply the idea of a topological quantum field theory (TQFT) to maps from manifolds into topological spaces. This leads to a notion of a $(d+1)$ -dimensional homotopy quantum field theory (HQFT) which may be described as a TQFT for closed d -dimensional manifolds and $(d+1)$ -dimensional cobordisms endowed with homotopy classes of maps into a given space. For a group π , we introduce cohomological HQFT's with target $K(\pi, 1)$ derived from cohomology classes of π and its subgroups of finite index. The main body of the paper is concerned with $(1+1)$ -dimensional HQFT's. We classify them in terms of so called crossed group-algebras. In particular, the cohomological $(1+1)$ -dimensional HQFT's over a field of characteristic 0 are classified by simple crossed group-algebras. We introduce two state sum models for $(1+1)$ -dimensional HQFT's and prove that the resulting HQFT's are direct sums of rescaled cohomological HQFT's. We also discuss a version of the Verlinde formula in this setting.

Introduction

Topological quantum field theories (TQFT's) produce topological invariants of manifolds using ideas from quantum field theory. For $d \geq 0$, a $(d+1)$ -dimensional TQFT over a commutative ring K assigns to every closed oriented d -dimensional manifold M a K -module A_M and assigns to every compact oriented $(d+1)$ -dimensional cobordism (W, M_0, M_1) a K -homomorphism $\tau(W) : A_{M_0} \rightarrow A_{M_1}$. These modules and homomorphisms should satisfy a few axioms, the main axiom being the multiplicativity of τ with respect to gluing of cobordisms. The study of TQFT's has been especially successful in low dimensions $d = 1$ and $d = 2$. Deep algebraic theories come up in both cases. The $(1+1)$ -dimensional TQFT's bijectively correspond to finite-dimensional commutative Frobenius algebras (see [Di], [Du]). The $(2+1)$ -dimensional TQFT's are closely related to quantum groups and braided categories (see [Tu]).

In this paper we apply the basic ideas of a TQFT to maps from manifolds into topological spaces. This suggests a notion of a $(d+1)$ -dimensional homotopy quantum field theory (HQFT) which may be briefly described as a TQFT for closed d -dimensional manifolds and $(d+1)$ -dimensional cobordisms endowed with homotopy classes of maps into a pointed path-connected space X . The $(0+1)$ -dimensional HQFT's over a field correspond bijectively to finite-dimensional representations of $\pi_1(X)$ or, equivalently, to finite-dimensional flat vector bundles over X . Thus, HQFT's may be viewed as higher-dimensional versions of flat vector bundles. The standard notion of a TQFT may be interpreted in this language as an HQFT with contractible target.

In the case where X is an Eilenberg-MacLane space $K(\pi, 1)$ corresponding to a group π , the homotopy classes of maps to X classify principal π -bundles. Thus, a $(d+1)$ -dimensional HQFT with target X yields invariants of principal π -bundles over closed d -dimensional manifolds and $(d+1)$ -dimensional cobordisms.

In the first part of the paper we discuss a general setting of HQFT's. In particular we introduce cohomological HQFT's determined by cohomology classes of groups. For $d = 1$, we define a wider class of semi-cohomological HQFT's.

In the second part of the paper we focus on algebraic structures underlying $(1+1)$ -dimensional HQFT's. For a group π , we introduce a notion of a π -algebra. Briefly speaking, this is an associative algebra L endowed with a splitting $L = \bigoplus_{\alpha \in \pi} L_\alpha$ such that $L_\alpha L_\beta \subset L_{\alpha\beta}$ for any $\alpha, \beta \in \pi$. We introduce various classes of such group-algebras including so-called crossed, semisimple, biangular, and non-degenerate group-algebras. The crossed group-algebras play the key role in the study of $(1+1)$ -dimensional HQFT's. Our main result is that each $(1+1)$ -dimensional HQFT with target X has an underlying crossed $\pi_1(X)$ -algebra and, moreover, this establishes a bijection between (the isomorphism classes of) $(1+1)$ -dimensional HQFT's with target X and crossed $\pi_1(X)$ -algebras. For $\pi = 1$, we obtain the well known equivalence between the $(1+1)$ -dimensional TQFT's and commutative Frobenius algebras (see [Di], [Du]).

The semi-cohomological $(1+1)$ -dimensional HQFT's (arising from 2-cohomology of groups) seem to be especially important since they satisfy a version of the Verlinde formula. The underlying crossed group-algebra of a semi-cohomological $(1+1)$ -dimensional HQFT is semisimple; we establish the converse provided the ground ring K is a field of characteristic 0.

In the third part of the paper we discuss lattice models for $(1+1)$ -dimensional HQFT's with target $K(\pi, 1)$. We introduce two lattice models derived from biangular (resp. non-degenerate) π -algebras. The first model generalizes the well known lattice model for $(1+1)$ -dimensional TQFT's, see [BP], [FHK]. We first present a map from a surface to $K(\pi, 1)$ by a π -system, i.e., a system of elements of π associated with 1-cells of a CW-decomposition of the surface. We fix a biangular π -algebra and use it to define a partition function (or a state sum) of each π -system. This partition function is homotopy invariant and determines a $(1+1)$ -dimensional HQFT's with target $K(\pi, 1)$. The model based on a non-degenerate π -algebra is more subtle: generally speaking, the corresponding partition functions are not homotopy invariant. To obtain an HQFT we sum up the partition functions over all π -systems in a given homotopy class.

We prove that the lattice $(1+1)$ -dimensional HQFT's over an algebraically closed field of characteristic 0 are semi-cohomological. This implies that these HQFT's satisfy the Verlinde formula.

This paper is a preliminary step towards a study of similar phenomena for $d = 2$ and $d = 3$.

Throughout the paper, the symbol K denotes a commutative ring with unit. The symbol π denotes a group.

Part I. Homotopy quantum field theories

1. Basic definitions and examples

1.1. Preliminaries. We shall use the language of pointed homotopy theory. A topological space is *pointed* if all its connected components are provided with base points. A map between pointed spaces is a continuous map sending base points into base points. Homotopies of such maps are always supposed to be constant on the base points. We shall work in the topological category although all our definitions apply in the smooth and piecewise-linear categories. Thus, by manifolds we shall mean topological manifolds.

Let X be a path-connected topological space with base point $x \in X$. We call an X -manifold any pair (a pointed closed oriented manifold M , a map $g_M : M \rightarrow X$). The map g_M is called the *characteristic map*. It sends the base points of all components of M into x . It is clear that a disjoint union of X -manifolds is an X -manifold. An empty set \emptyset is considered as an X -manifold of any given dimension. An X -homeomorphism of X -manifolds $f : M \rightarrow M'$ is an orientation preserving homeomorphism sending the base points of M onto those of M' and such that $g_M = g_{M'} \circ f$ where $g_M, g_{M'}$ are the characteristic maps of M, M' , respectively.

By a cobordism we shall mean a triple (W, M_0, M_1) where W is a compact oriented manifold whose boundary is a disjoint union of pointed closed oriented manifolds M_0, M_1 such that the orientation of M_1 (resp. M_0) is induced by the one of W (resp. is opposite to the one induced from W). The manifold W itself is not supposed to be pointed.

An X -cobordism is a cobordism (W, M_0, M_1) endowed with a map $W \rightarrow X$ sending the base points of the boundary components into x . Both bases M_0 and M_1 are considered as X -manifolds with characteristic maps obtained by restricting the given map $W \rightarrow X$. If (W, M_0, M_1) is an X -cobordism, then $(-W, M_1, M_0)$ is also an X -cobordism where $-W$ denotes W with opposite orientation. An X -homeomorphism of X -cobordisms $f : (W, M_0, M_1) \rightarrow (W', M'_0, M'_1)$ is an orientation preserving homeomorphism inducing X -homeomorphisms $M_0 \rightarrow M'_0, M_1 \rightarrow M'_1$ and such that $g_W = g_{W'} \circ f$ where $g_W, g_{W'}$ are the characteristic maps of W, W' , respectively.

We can glue X -cobordisms along the bases. If $(W_0, M_0, N), (W_1, N', M_1)$ are X -cobordisms and $f : N \rightarrow N'$ is an X -homeomorphism then the gluing of W_0 to W_1 along f yields a new X -cobordism with bases M_0 and M_1 . Here it is essential that $g_N = g_{N'} \circ f$.

1.2. Definition of HQFT's. Fix an integer $d \geq 0$ and a path-connected topological space X with base point $x \in X$. We define a $(d+1)$ -dimensional homotopy quantum field theory (A, τ) with target X . It will take values in the category of projective K -modules of finite type (= direct summands of K^n with $n = 0, 1, \dots$). The reader may restrict himself/herself to the case where K is a field so that projective K -modules of finite type are just finite-dimensional vector spaces over K .

A $(d+1)$ -dimensional HQFT (A, τ) with target X assigns a projective K -module of finite type A_M to any d -dimensional X -manifold M , a K -isomorphism

$f_{\#} : A_M \rightarrow A_{M'}$ to any X -homeomorphism of d -dimensional X -manifolds $f : M \rightarrow M'$, and a K -homomorphism $\tau(W) : A_{M_0} \rightarrow A_{M_1}$ to any $(d+1)$ -dimensional X -cobordism (W, M_0, M_1) . These modules and homomorphisms should satisfy the following eight axioms.

(1.2.1) For any X -homeomorphisms of d -dimensional X -manifolds $f : M \rightarrow M', f' : M' \rightarrow M''$, we have $(f'f)_{\#} = f'_{\#}f_{\#}$. The isomorphism $f_{\#} : A_M \rightarrow A_{M'}$ is invariant under isotopies of f in the class of X -homeomorphisms.

(1.2.2) For any disjoint d -dimensional X -manifolds M, N , there is a natural isomorphism $A_{M \amalg N} = A_M \otimes A_N$ where \otimes is the tensor product over K .

(1.2.3) $A_{\emptyset} = K$.

(1.2.4) The homomorphism τ associated with X -cobordisms is natural with respect to X -homeomorphisms.

(1.2.5) If a $(d+1)$ -dimensional X -cobordism W is a disjoint union of two X -cobordisms W_1, W_2 then $\tau(W) = \tau(W_1) \otimes \tau(W_2)$.

(1.2.6) If an X -cobordism (W, M_0, M_1) is obtained from two $(d+1)$ -dimensional X -cobordisms (W_0, M_0, N) and (W_1, N', M_1) by gluing along an X -homeomorphism $f : N \rightarrow N'$ then

$$\tau(W) = \tau(W_1) \circ f_{\#} \circ \tau(W_0) : A_{M_0} \rightarrow A_{M_1}.$$

(1.2.7) For any d -dimensional X -manifold $(M, g : M \rightarrow X)$ and any map $F : M \times [0, 1] \rightarrow X$ such that $F|_{M \times 0} = F|_{M \times 1} = g$ and $F(m \times [0, 1]) = x$ for all base points $m \in M$, we have $\tau(M \times [0, 1], F) = \text{id} : A_M \rightarrow A_M$ where the cylinder $M \times [0, 1]$ is viewed as an X -cobordism with bases $M \times 0 = M, M \times 1 = M$ and characteristic map F .

(1.2.8) For any $(d+1)$ -dimensional X -cobordism $W = (W, g : W \rightarrow X)$, the homomorphism $\tau(W)$ is preserved under any homotopy of g relative to ∂W .

Axioms (1.2.1) - (1.2.7) form a version of the standard definition of a TQFT, cf. [At], [Tu, Chapter III]. It is sometimes convenient to consider the homomorphism $\tau(W)$ associated to an X -cobordism (W, M_0, M_1) as a vector

$$\tau(W) \in \text{Hom}_K(A_{M_0}, A_{M_1}) = A_{M_0}^* \otimes A_{M_1}.$$

In this language, axiom (1.2.6) says that $\tau(W)$ is obtained from $\tau(W_0) \otimes \tau(W_1)$ by the tensor contraction induced by the pairing $a \otimes b \mapsto b(f_{\#}(a)) : A_N \otimes A_{N'}^* \rightarrow K$.

Any closed oriented $(d+1)$ -dimensional manifold W endowed with a map $g : W \rightarrow X$ can be considered as a cobordism with empty bases. The corresponding K -linear endomorphism of $A_{\emptyset} = K$ is multiplication by a certain $\tau(W) \in K$. By (1.2.8), $\tau(W)$ is a homotopy invariant of g . More generally, the modules and homomorphisms provided by any HQFT depend only on the homotopy classes of the characteristic maps, see Section 2.1.

We define a few simple operations on $(d+1)$ -dimensional HQFT's with target X . The direct sum of HQFT's $(A, \tau) \oplus (A', \tau')$ is defined by $(A \oplus A')_M = A_M \oplus A'_M$ and $(\tau \oplus \tau')(W) = \tau(W) \oplus \tau'(W)$. The tensor product is defined similarly using \otimes instead of \oplus . The dual (A^*, τ^*) of an HQFT (A, τ) is defined by $A_M^* =$

$\text{Hom}_K(A_M, K)$ for any X -manifold M with action of X -homeomorphisms obtained by transposition from the one given by (A, τ) . We define $\tau^*(W) : A_{M_0}^* \rightarrow A_{M_1}^*$ as the transpose of $\tau(-W) : A_{M_1} \rightarrow A_{M_0}$. All the axioms of an HQFT are straightforward.

We define a category denoted $Q_{d+1}(X, x)$ (or shorter $Q_{d+1}(X)$) whose objects are $(d+1)$ -dimensional HQFT's with target X . A morphism $(A, \tau) \rightarrow (A', \tau')$ in this category is a family of K -homomorphisms $\{\rho_M : A_M \rightarrow A'_M\}_M$ where M runs over d -dimensional X -manifolds such that: $\rho_\emptyset = \text{id}_K$; for disjoint X -manifolds M, N , we have $\rho_{M \amalg N} = \rho_M \otimes \rho_N$; the natural square diagrams associated with homeomorphisms of X -manifolds and with X -cobordisms are commutative. It can be shown (we shall not use it) that all morphisms in the category $Q_{d+1}(X, x)$ are isomorphisms.

It is easy to deduce from definitions that the isomorphism classes of $(0+1)$ -dimensional HQFT's with target X correspond bijectively to the isomorphism classes of linear actions of $\pi_1(X, x)$ on projective K -modules of finite type. Under this correspondence, the invariant τ of a map $g : S^1 \rightarrow X$ equals the trace of the conjugacy class of linear endomorphisms determined by g .

If the space X consists of only one point x then all references to maps into X are redundant and we obtain the usual definition of a topological quantum field theory (TQFT) for pointed closed oriented d -dimensional manifolds and their cobordisms. Restricting any HQFT (A, τ) with target (X, x) to those X -manifolds and X -cobordisms whose characteristic map is a constant map into $x \in X$, we obtain an underlying TQFT of (A, τ) .

1.3. Primitive cohomological HQFT's. Let X be a Eilenberg-MacLane space of type $K(\pi, 1)$ where π is a group. (Speaking about Eilenberg-MacLane spaces we always assume that they are CW-complexes). Recall that the symbol K^* denotes the multiplicative group consisting of the invertible elements of K . For each $\theta \in H^{d+1}(X; K^*) = H^{d+1}(\pi; K^*)$, we shall define a $(d+1)$ -dimensional HQFT (A, τ) with target X called the *primitive cohomological HQFT* associated with θ and denoted (A^θ, τ^θ) . This construction is inspired by the work of Freed and Quinn [FQ] on TQFT's associated with finite groups.

Choose a singular $(d+1)$ -dimensional cocycle on X with values in K^* representing θ . By abuse of notation we denote this cocycle by the same symbol θ . Let M be a d -dimensional X -manifold. Then A_M is a free K -module of rank 1 defined as follows. A d -dimensional singular cycle $a \in C_d(M) = C_d(M; \mathbf{Z})$ is said to be *fundamental* if it represents the fundamental class $[M] \in H_d(M; \mathbf{Z})$ defined as the sum of the fundamental classes of the components of M . Every fundamental cycle $a \in C_d(M)$ determines a non-zero element $\langle a \rangle \in A_M$. If $a, b \in C_d(M)$ are two fundamental cycles, then we impose the equality $\langle a \rangle = g^*(\theta)(c)\langle b \rangle$ where $g : M \rightarrow X$ is the characteristic map of M and c is a $(d+1)$ -dimensional singular chain in M such that $\partial c = a - b$. Note that $g^*(\theta)(c) \in K^*$ does not depend on the choice of c : if $\partial c = \partial c'$ with $c, c' \in C_{d+1}(M)$ then $c - c' = \partial e$ with $e \in C_{d+2}(M)$ and

$$g^*(\theta)(c)/g^*(\theta)(c') = g^*(\theta)(\partial e) = \theta(\partial g_*(e)) = \partial \theta(g_*(e)) = 1.$$

It is easy to check that A_M is a well defined free K -module of rank 1. An X -homeomorphism of d -dimensional X -manifolds $f : M \rightarrow M'$ induces an isomorphism $A_M \rightarrow A_{M'}$ sending the generator $\langle a \rangle \in A_M$ as above into $\langle f_*(a) \rangle \in A_{M'}$.

Consider a $(d+1)$ -dimensional X -cobordism (W, M_0, M_1) . Let $B \in C_{d+1}(W) = C_{d+1}(W; \mathbf{Z})$ be a fundamental chain, i.e., a singular chain representing the fundamental class $[W] \in H_{d+1}(W, \partial W; \mathbf{Z})$ defined as the sum of the fundamental classes of the components of W . It is clear that $\partial B = -b_0 + b_1$ where b_0, b_1 are fundamental cycles of M_0, M_1 , respectively. We define $\tau(W) : A_{M_0} \rightarrow A_{M_1}$ by

$$\tau(W)(\langle b_0 \rangle) = (g^*(\theta)(B))^{-1} \langle b_1 \rangle$$

where $g : W \rightarrow X$ is the characteristic map of W . Let us verify that $\tau(W)$ does not depend on the choice of B . Let $B' \in C_{d+1}(W)$ be another fundamental chain with $\partial B' = -b'_0 + b'_1$ where b'_i is a fundamental cycle of M_i with $i = 0, 1$. Choose $c_i \in C_{d+1}(M_i)$ such that $\partial c_i = b_i - b'_i$ for $i = 0, 1$. By definition, $\langle b_i \rangle = g^*(\theta)(c_i) \langle b'_i \rangle$ where $g_i : M_i \rightarrow X$ is the characteristic map of M_i . To see that $\tau(W)$ is well defined it suffices to check the equality in K^*

$$(1.3.a) \quad g^*(\theta)(c_0)(g^*(\theta)(B'))^{-1} = g^*(\theta)(c_1)(g^*(\theta)(B))^{-1}.$$

Clearly, $B + c_0 - B' - c_1 \in C_{d+1}(W)$ is a cycle representing 0 in $H_{d+1}(W, \partial W; \mathbf{Z})$. Note that the inclusion $H_{d+1}(W; \mathbf{Z}) \rightarrow H_{d+1}(W, \partial W; \mathbf{Z})$ is injective. Therefore the cycle $B + c_0 - B' - c_1$ is a boundary in W . This implies (1.3.a).

It remains to verify the axioms of an HQFT. Axioms (1.2.1) - (1.2.5) are straightforward. Let us check (1.2.6). Let (W, M_0, M_1) be a $(d+1)$ -dimensional X -cobordism obtained from two X -cobordisms (W_0, M_0, N) and (W_1, N', M_1) by gluing along an X -homeomorphism $f : N \rightarrow N'$. Let $B_0 \in C_{d+1}(W_0)$ be a fundamental chain with $\partial B_0 = -b_0 + b$ where b_0, b are fundamental cycles of M_0, N , respectively. Clearly, $f_*(b)$ is a fundamental cycle of N' . Choose a fundamental chain $B_1 \in C_{d+1}(W_1)$ such that $\partial B_1 = -f_*(b) + b_1$ where b_1 is a fundamental cycle of M_1 . By definition, the composition $\tau(W_1) \circ f_{\#} \circ \tau(W_0)$ sends the generator $\langle b_0 \rangle$ of A_{M_0} into $(g_0^*(\theta)(B_0)g_1^*(\theta)(B_1))^{-1} \langle b_1 \rangle$ where g_j is the characteristic map of W_j for $j = 0, 1$. Observe that under the gluing of W_0 to W_1 along $f : N \rightarrow N'$, the chain $B_0 + B_1$ is mapped into a fundamental chain $B \in C_{d+1}(W)$ with $\partial B = -b_0 + b_1$. Therefore, $g_0^*(\theta)(B_0)g_1^*(\theta)(B_1) = g^*(\theta)(B)$ where $g : W \rightarrow X$ is the characteristic map of W . By definition,

$$\tau(W)(\langle b_0 \rangle) = (g^*(\theta)(B))^{-1} \langle b_1 \rangle = (\tau(W_1) \circ f_{\#} \circ \tau(W_0))(\langle b_0 \rangle).$$

Let us check (1.2.7). It is here that we use the fact that we work in the pointed category and that $X = K(\pi, 1)$. Consider a d -dimensional X -manifold $(M, g : M \rightarrow X)$ and a map $F : M \times [0, 1] \rightarrow X$ such that $F|_{M \times 0} = F|_{M \times 1} = g$ and $F(m \times [0, 1]) = x$ for all base points m of the components of M . We choose a fundamental chain $B \in C_{d+1}(M \times [0, 1])$ so that $\partial B = -(b \times 0) + (b \times 1)$ where b is a fundamental cycle of M . By definition, the homomorphism $\tau(M \times [0, 1], F)$

sends the generator $\langle b \rangle \in A_M$ into $(F^*(\theta)(B))^{-1}\langle b \rangle$. It remains to prove that $F^*(\theta)(B) = 1$.

Consider the map $p : M \times [0, 1] \rightarrow M \times S^1$ obtained by the gluing $M \times 0 = M = M \times 1$. It is clear that $p_*(B)$ is a fundamental cycle in $M \times S^1$. The map F induces a map $\tilde{F} : M \times S^1 \rightarrow X$ such that $F = \tilde{F} \circ p$. Therefore $F^*(\theta)(B) = \theta(F_*(B)) = \theta(\tilde{F}_*([M \times S^1]))$ where $\tilde{F}_*([M \times S^1]) \in H_{d+1}(X; \mathbf{Z})$. We claim that the latter homology class is trivial. This would imply the equality $F^*(\theta)(B) = 1$. To prove our claim it suffices to observe that the map \tilde{F} extends to a map $M \times D^2 \rightarrow X$ where D^2 is a 2-disc bounded by S^1 . This follows easily from the assumptions that $X = K(\pi, 1)$ and each component of M contains a point m such that $\tilde{F}(m \times S^1) = x$.

Let us verify (1.2.8). Let $(W, g : W \rightarrow X)$ be an X -cobordism of dimension $d + 1$. We should verify that for a fundamental chain $B \in C_{d+1}(W)$, the element $g^*(\theta)(B) \in K^*$ is preserved under any homotopy of g relative to ∂W . Consider the manifold \tilde{W} obtained from $W \times [0, 1]$ by contracting each interval $w \times [0, 1]$ with $w \in \partial W$ to a point. The homotopy of g induces a map $\tilde{g} : \tilde{W} \rightarrow X$. The manifold $\partial \tilde{W}$ is obtained by gluing $W \times 0$ to $W \times 1$ along $\partial W \times 0 = \partial W = \partial W \times 1$. The chain $(B \times 0) - (B \times 1)$ in $\partial \tilde{W}$ represents the fundamental class of $\partial \tilde{W}$ and therefore bounds a singular chain in \tilde{W} . This implies that $\tilde{g}^*(\theta)(B \times 0) = \tilde{g}^*(\theta)(B \times 1)$. This is exactly the equality we need.

It is easy to compute the element $\tau(W, g) \in K$ associated by this HQFT with a (closed oriented) $(d + 1)$ -dimensional X -manifold $(W, g : W \rightarrow X)$:

$$\tau(W, g) = g^*(\theta)([W]) = \theta(g_*([W])) \in K^*.$$

Note finally that the primitive cohomological HQFT's arising as above from different singular cocycles representing $\theta \in H^{d+1}(X; K^*) = H^{d+1}(\pi; K^*)$ are isomorphic. Therefore the isomorphism class of these HQFT's depends only on θ . It is easy to describe explicitly the primitive cohomological HQFT associated with $0 \in H^{d+1}(\pi; K^*)$: this HQFT assigns K to all d -dimensional X -manifolds and assigns $\text{id}_K : K \rightarrow K$ to all X -homeomorphisms and to all $(d + 1)$ -dimensional X -cobordisms.

1.4. Rescaling of HQFT's. Further examples of HQFT's can be obtained from additive invariants of X -cobordisms. An integer valued function ρ of $(d + 1)$ -dimensional X -cobordisms is an *additive invariant* if it is preserved under X -homeomorphisms and homotopies of the characteristic map and is additive under the gluing of X -cobordisms described in Section 1.1. Fix $k \in K^*$. Using an additive invariant ρ , we can transform any $(d + 1)$ -dimensional HQFT (A, τ) into a k^ρ -rescaled HQFT which coincides with (A, τ) except that the homomorphism associated with a $(d + 1)$ -dimensional X -cobordism (W, M_0, M_1) is equal to $k^{\rho(W)}\tau(W)$.

For any $d \geq 0$, an additive invariant of a $(d + 1)$ -dimensional X -cobordism (W, M_0, M_1) is given by $\rho(W) = \chi(W, M_0)$ where χ is the Euler characteristic. This example can be refined using the semi-characteristic. Consider for concrete-

ness the case $d = 1$. Set

$$(1.4.a) \quad \rho_0(W) = (\chi(W) + b_0(M_0) - b_0(M_1))/2 \in \mathbf{Z}$$

where $b_0(M)$ is the number of components of M . It is obvious that ρ_0 is an additive invariant of 2-dimensional X -cobordisms.

For $d \equiv 0 \pmod{3}$, examples of additive invariants are provided by the signature of W and the G -signatures of W determined by the homomorphism $\pi_1(W) \rightarrow \pi_1(X)$ induced by the characteristic map $W \rightarrow X$ and a fixed homomorphism from $\pi_1(X)$ into a finite group G .

1.5. Transfer. Let X be a connected CW-complex with base point $x \in X$. Let $p : E \rightarrow X$ be a connected finite-sheeted covering with base point $e \in p^{-1}(x)$. From any $(d+1)$ -dimensional HQFT (A, τ) with target E we shall derive a $(d+1)$ -dimensional HQFT $(\tilde{A}, \tilde{\tau})$ with target X . It is called the *transfer* of (A, τ) .

We first fix a map $q : E \rightarrow E$ which is homotopic to the identity and sends the set $p^{-1}(x)$ into the point e . (To construct such a map q one may choose for each $y \in p^{-1}(x)$ a path from y to e in E and then push all $y \in p^{-1}(x)$ into e along these paths. This extends to a homotopy of the identity map into q .) Let $(M, g : M \rightarrow X)$ be a d -dimensional X -manifold. There is a finite number of lifts of g to E , i.e., of maps $\tilde{g} : M \rightarrow E$ such that $p\tilde{g} = g$. Note that each pair $(M, q\tilde{g})$ is an E -manifold. Consider the K -module

$$\tilde{A}_M = \bigoplus_{\tilde{g}, p\tilde{g}=g} A_{(M, q\tilde{g})}.$$

Any X -homeomorphism of d -dimensional X -manifolds $f : (M, g) \rightarrow (M', g')$ induces a K -isomorphism $f_{\#} : \tilde{A}_{(M, g)} \rightarrow \tilde{A}_{(M', g')}$ as follows. Since $gf = g'$, any lift $\tilde{g} : M \rightarrow E$ of g induces a lift $\tilde{g}f : M' \rightarrow E$ of g' . The HQFT (A, τ) yields an isomorphism $f_{\#} : A_{(M, q\tilde{g})} \rightarrow A_{(M', q\tilde{g}f)}$. The direct sum of these isomorphisms yields the desired isomorphism $f_{\#} : \tilde{A}_{(M, g)} \rightarrow \tilde{A}_{(M', g')}$.

Now, consider a $(d+1)$ -dimensional X -cobordism (W, M_0, M_1) with characteristic map $g : W \rightarrow X$. As above, there is a finite number of lifts $\tilde{g} : W \rightarrow E$ such that $p\tilde{g} = g$. Each such lift \tilde{g} can be restricted to the bases of W and induces in this way certain lifts, \tilde{g}_0, \tilde{g}_1 , of the characteristic maps $M_0 \rightarrow X, M_1 \rightarrow X$. Then the HQFT (A, τ) yields a homomorphism

$$\tau(W, q\tilde{g}) : A_{(M_0, q\tilde{g}_0)} \rightarrow A_{(M_1, q\tilde{g}_1)}.$$

The sum of these homomorphisms corresponding to all \tilde{g} defines a homomorphism $\tilde{\tau}(W, g) : \tilde{A}_{M_0} \rightarrow \tilde{A}_{M_1}$. (Warning: a lift of the characteristic map $M_0 \rightarrow X$ to E may extend to different lifts of g so that the sum in question is in general not a direct sum.) It is easy to verify that these definitions yield a $(d+1)$ -dimensional HQFT $(\tilde{A}, \tilde{\tau})$ with target X . In particular, for a (closed oriented) $(d+1)$ -dimensional X -manifold $(W, g : W \rightarrow X)$,

$$\tilde{\tau}(W, g) = \sum_{\tilde{g} : W \rightarrow E, p\tilde{g}=g} \tau(W, q\tilde{g}) \in K.$$

We leave it as an exercise for the reader to check that the isomorphism class of the HQFT $(\tilde{A}, \tilde{\tau})$ does not depend on the choice of the map q (cf. Section 2.3).

1.6. Cohomological and semi-cohomological HQFT's. Let π be a group and X be an Eilenberg-MacLane space of type $K(\pi, 1)$ with base point x . Let $G \subset \pi$ be a subgroup of finite index n and $p : E \rightarrow X$ be the connected n -sheeted covering of X corresponding to G with base point $e \in p^{-1}(x)$. For each $\theta \in H^{d+1}(G; K^*) = H^{d+1}(E; K^*)$, the transfer transforms the primitive cohomological HQFT with target $E = K(G, 1)$ associated with θ into a $(d+1)$ -dimensional HQFT with target X called the *cohomological HQFT* associated with θ and denoted by $(A^{\pi, G, \theta}, \tau^{\pi, G, \theta})$. It follows from definitions that for a connected d -dimensional X -manifold $(M, g : M \rightarrow X)$ with base point $m \in M$, the K -module $A_M^{\pi, G, \theta}$ is free of rank

$$\text{card}\{i \in G \backslash \pi \mid \text{Im}(g_{\#} : \pi_1(M, m) \rightarrow \pi_1(X, x)) \subset i^{-1}Gi\}$$

(this rank does not depend on θ). To compute the invariant $\tau^{\pi, G, \theta}(W)$ for a (closed oriented) connected $(d+1)$ -dimensional X -manifold $(W, g : W \rightarrow X)$, we first deform g so that it sends a point $w \in W$ into x . For each right coset class $i \in G \backslash \pi$, choose a representative $\omega_i \in i$ so that $i = G\omega_i \subset \pi$. Then

$$\tau^{\pi, G, \theta}(W) = \sum_{i \in G \backslash \pi, g_{\#}(\pi_1(W, w)) \subset i^{-1}Gi} g_i^*(\theta)([W]) \in K$$

where $g_i : (W, w) \rightarrow (E, e)$ is a map inducing the homomorphism $\omega_i g_{\#} \omega_i^{-1}$ of the fundamental groups. (The value $g_i^*(\theta)([W]) \in K^*$ does not depend on the choice of ω_i .)

We define semi-cohomological HQFT's as direct sums of rescaled cohomological HQFT's. Specifically, a $(1+1)$ -dimensional HQFT with target $X = K(\pi, 1)$ is said to be *semi-cohomological* if it splits as a finite direct sum $\oplus_i (A_i, \tau_i)$ where each (A_i, τ_i) is a $(1+1)$ -dimensional HQFT with target X obtained by $k_i^{\rho_0}$ -rescaling from a cohomological HQFT $(A^{\pi, G_i, \theta_i}, \tau^{\pi, G_i, \theta_i})$ where $k_i \in K^*$, ρ_0 is the additive invariant of 2-dimensional cobordisms defined by (1.4.a), $G_i \subset \pi$ is a subgroup of finite index, and $\theta_i \in H^2(G_i; K^*)$.

1.7. Hermitian and unitary HQFT's. Assume that the ground ring K has a ring involution $k \mapsto \bar{k} : K \rightarrow K$. Let $d \geq 0$ and (A, τ) be a $(d+1)$ -dimensional HQFT with target a pointed path-connected space X . A Hermitian structure on (A, τ) assigns to each d -dimensional X -manifold M a non-degenerate Hermitian pairing $\langle \cdot, \cdot \rangle_M : A_M \otimes_K A_M \rightarrow K$ satisfying the following two conditions.

(1.7.1) The pairing $\langle \cdot, \cdot \rangle_M$ is natural with respect to X -homeomorphisms and multiplicative with respect to disjoint union; for $M = \emptyset$ the pairing $\langle \cdot, \cdot \rangle_M$ is determined by the unit 1×1 -matrix.

(1.7.2) For any X -cobordism (W, M_0, M_1) and any $a \in A_{M_0}, b \in A_{M_1}$, we have

$$(1.7.a) \quad \langle \tau(W)(a), b \rangle_{M_1} = \langle a, \tau(-W)(b) \rangle_{M_0}.$$

An HQFT with a Hermitian structure is called a Hermitian HQFT. If $K = \mathbf{C}$ with usual complex conjugation and the Hermitian form $\langle \cdot, \cdot \rangle_M$ is positive definite for every M , we say that the Hermitian HQFT is unitary.

Note two properties of a Hermitian HQFT (A, τ) . First, if W is a $(d+1)$ -dimensional closed X -manifold then $\tau(-W) = \overline{\tau(W)}$. Secondly, the group of X -self-homeomorphisms of any d -dimensional closed X -manifold M acts in A_M preserving the Hermitian form $\langle \cdot, \cdot \rangle_M$. For a unitary HQFT we obtain a unitary action. This implies in particular the following estimate for the value of τ on the mapping torus W_g of an X -homeomorphism $g : M \rightarrow M$: if (A, τ) is a unitary HQFT then $|\tau(W_g)| \leq \dim_{\mathbf{C}} A_M$ (cf. [Tu], Chapter III).

It is easy to check that direct sums, tensor products, and transfers of Hermitian (resp. unitary) HQFT's are again Hermitian (resp. unitary) HQFT's. We can define a category, $HQ_{d+1}(X)$ (resp. $UQ_{d+1}(X)$), whose objects are $(d+1)$ -dimensional Hermitian (resp. unitary) HQFT's with target X . The morphisms in this category are defined as in Section 1.2 with additional requirement that the homomorphisms $\{\rho_M\}_M$ preserve the Hermitian pairing.

The construction of primitive cohomological HQFT's can be refined to yield Hermitian and unitary HQFT's. Consider the multiplicative group $S = \{k \in K^* \mid k\bar{k} = 1\} \subset K^*$. For each $\theta \in H^{d+1}(\pi; S)$, the $(d+1)$ -dimensional HQFT (A^θ, τ^θ) defined in Section 1.3 can be provided with a Hermitian structure as follows. For a d -dimensional X -manifold M (where $X = K(\pi, 1)$) and a fundamental cycle $a \in C_d(M)$, set $\langle \langle a \rangle, \langle a \rangle \rangle_M = 1 \in K$ where $\langle a \rangle \in A_M$ is the vector represented by a . This yields a well defined Hermitian form on $A_M = K\langle a \rangle$ satisfying (1.7.1) and (1.7.2). The isomorphism class of the resulting Hermitian HQFT depends only on θ . If $K = \mathbf{C}$ then $S = S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$ and the Hermitian HQFT determined by any $\theta \in H^{d+1}(\pi; S^1)$ is unitary.

Note that the isomorphism classes of $(0+1)$ -dimensional unitary HQFT's with target X correspond bijectively to the isomorphism classes of finite-dimensional flat unitary bundles over X .

2. Homotopy properties of HQFT's

2.1. Homotopy invariance of HQFT's. It was already mentioned above that the modules and homomorphisms provided by an HQFT depend only on the homotopy classes of the characteristic maps. Here we discuss this in detail.

Fix a path-connected topological space X with base point x . Fix a $(d+1)$ -dimensional HQFT (A, τ) with target X . For a pointed space Y , we shall denote the set of homotopy classes of maps (in the category of pointed spaces) $Y \rightarrow X$ by $\text{Map}(Y, X)$.

We introduce homotopy X -manifolds exactly as X -manifolds with the difference that instead of maps to X we speak of homotopy classes of maps. Thus, a homotopy X -manifold is a pair (a pointed closed oriented manifold M , an element $G \in \text{Map}(M, X)$). For a homotopy X -manifold (M, G) we define a K -module $A_{(M, G)}$ as follows. Observe that any homotopy $F : M \times [0, 1] \rightarrow X$ between two maps $g, g' : M \rightarrow X$ belonging to the class G provides the cylinder $M \times [0, 1]$

with the structure of an X -cobordism. This cobordism gives a homomorphism $\tau_F : A_{(M,g)} \rightarrow A_{(M,g')}$. It follows from (1.2.6) that $\tau_{FF'} = \tau_{F'}\tau_F$ for any composable homotopies F, F' . Axiom (1.2.7) implies that for any homotopy F_0 relating a map to itself, $\tau_{F_0} = \text{id}$. This implies that for any homotopy F between g and g' as above, the homomorphism τ_F is an isomorphism depending only on g, g' and independent of the choice of F . We identify the modules $\{A_{(M,g)} \mid g \in G\}$ along the isomorphisms $\{\tau_F\}_F$. This gives a K -module $A_{(M,G)}$ depending on (M, G) and canonically isomorphic to each $A_{(M,g)}$ with $g \in G$.

An X -homeomorphism of homotopy X -manifolds $f : (M, G) \rightarrow (M', G')$ is an orientation preserving homeomorphism $M \rightarrow M'$ sending the base points of M onto those of M' and such that $G = G'f$ where the equality is understood as an equality of homotopy classes. We define the action $f_\# : A_{(M,G)} \rightarrow A_{(M',G')}$ of f as the composition

$$A_{(M,G)} \xrightarrow{=} A_{(M,g'f)} \xrightarrow{f_\#} A_{(M',g')} \xrightarrow{=} A_{(M',G')}$$

where $g' : M' \rightarrow X$ is a map representing G' and the first and third homomorphisms are the canonical identifications. We claim that this composition does not depend on the choice of $g' \in G'$. Let $g'' : M' \rightarrow X$ be another representative of the class G' and let F' be a homotopy between g' and g'' . It is clear that F' induces a homotopy $F = F' \circ (f \times \text{id}_{[0,1]})$ between $g'f$ and $g''f$. Axiom (1.2.4) implies that the diagram

$$\begin{array}{ccc} A_{(M,g'f)} & \xrightarrow{f_\#} & A_{(M',g')} \\ \tau_F \downarrow & & \downarrow \tau_{F'} \\ A_{(M,g''f)} & \xrightarrow{f_\#} & A_{(M',g'')} \end{array}$$

is commutative. This implies our claim.

We define a homotopy X -cobordism as a cobordism (W, M_0, M_1) endowed with a homotopy classes of maps $W \rightarrow X$. (The maps should send the base points of ∂W into x and the homotopies should be constant on the base points; we denote the set of the corresponding homotopy classes by $\text{Map}(W, X)$). For a $(d+1)$ -dimensional homotopy X -cobordism $(W, M_0, M_1, G \in \text{Map}(W, X))$ we shall define a homomorphism $\tau(W, G) : A_{(M_0, G_0)} \rightarrow A_{(M_1, G_1)}$ where $G_j \in \text{Map}(M_j, X)$ is the restriction of G to M_j , for $j = 0, 1$. Let $g : W \rightarrow X$ be a representative of the homotopy class G . Let $g_j \in G_j$ be the restriction of g to M_j . We define $\tau(W, G)$ as the composition

$$A_{(M_0, G_0)} \xrightarrow{=} A_{(M_0, g_0)} \xrightarrow{\tau(W, g)} A_{(M_1, g_1)} \xrightarrow{=} A_{(M_1, G_1)}$$

where the first and third homomorphisms are the canonical identifications and the second homomorphism is determined by the X -cobordism (W, g) . We claim that $\tau(W, G)$ does not depend on the choice of g in the class G . Let $g' : W \rightarrow X$ be

another representative of G and let F be a homotopy between g and g' . We should prove the commutativity of the diagram

$$\begin{array}{ccc} A_{(M_0, g_0)} & \xrightarrow{\tau(W, g)} & A_{(M_1, g_1)} \\ \tau_{F_0} \downarrow & & \downarrow \tau_{F_1} \\ A_{(M_0, g'_0)} & \xrightarrow{\tau(W, g')} & A_{(M_1, g'_1)} \end{array}$$

where F_j denotes the restriction of F to $M_j \times [0, 1]$ for $j = 0, 1$. Let F_1^{-1} be the homotopy $M_1 \times [0, 1] \rightarrow X$ inverse to F_1 (i.e., $F_1^{-1}(a, t) = F_1(a, 1 - t)$ for any $a \in M_1, t \in [0, 1]$). Consider the X -cobordism W' obtained by gluing the X -cobordisms $(M_0 \times [0, 1], F_0)$, (W, g') , and $(M_1 \times [0, 1], F_1^{-1})$ along $M_0 \times 1 = M_0 \subset \partial W$ and $M_1 \times 0 = M_1 \subset \partial W$. By axiom (1.2.6), $\tau(W') = (\tau_{F_1})^{-1} \tau(W, g') \tau_{F_0}$. On the other hand, it is clear that W' is just the same cobordism W with another characteristic map to X . Moreover, the homotopy F induces a homotopy of this characteristic map into g relative to ∂W . By axiom (1.2.8), $\tau(W') = \tau(W, g)$. Hence the diagram above is commutative.

The constructions of this subsection show that from the very beginning we can formulate the definition of an HQFT in terms of the homotopy classes of characteristic maps. In the sequel we shall make no difference between characteristic maps and their homotopy classes.

We end this subsection with a simple but useful lemma. We say that two maps g, g' from a cobordism W to X are equivalent if they are homotopic (rel ∂W) in the complement of a small ball in $\text{Int}W$. It is easy to see that this is indeed an equivalence relation. Note that equivalent maps $W \rightarrow X$ coincide on ∂W .

2.1.1. Lemma. *Let $d \geq 1$ and (W, M_0, M_1) be a $(d+1)$ -dimensional cobordism. If $g, g' : W \rightarrow X$ are two equivalent maps then*

$$\tau(W, g) = \tau(W, g') : A_{(M_0, g|_{M_0})} \rightarrow A_{(M_1, g|_{M_1})}.$$

Proof. By (1.2.5), it suffices to prove the lemma for connected W . If $\partial W = \emptyset$ then we split W along a small embedded d -dimensional sphere into a union of two cobordisms. By (1.2.6), the claim for these smaller cobordisms would imply the claim for W . Thus, it suffices to consider the case where W is connected and $\partial W \neq \emptyset$. Suppose for concreteness that $M_0 \neq \emptyset$. By assumptions, $g' : W \rightarrow X$ is homotopic (rel ∂W) to a map $g'' : W \rightarrow X$ which coincides with g outside a $(d+1)$ -dimensional ball $B \subset W$. Choose a regular neighborhood $M_0 \times [0, 1] \subset W$ of M_0 such that $B \subset M_0 \times (0, 1)$ and for any base point m (of a component) of M_0 the arc $m \times [0, 1]$ is disjoint from B . Deforming if necessary g, g'' on $\text{Int}(W) \setminus B$, we can assume that $g(m \times [0, 1]) = g''(m \times [0, 1]) = x$ for any base point m of M_0 . By the argument given at the beginning of Section 2.1, $\tau(M_0 \times [0, 1], g) = \tau(M_0 \times [0, 1], g'')$.

The cobordism W is obtained by gluing $M_0 \times [0, 1]$ to $V = W \setminus (M_0 \times [0, 1])$ along $M_0 \times 1$. By (1.2.6), (1.2.8),

$$\begin{aligned}\tau(W, g') &= \tau(W, g'') = \tau(V, g'') \tau(M_0 \times [0, 1], g'') \\ &= \tau(V, g) \tau(M_0 \times [0, 1], g) = \tau(W, g).\end{aligned}$$

2.2. Functoriality. The HQFT's can be pulled back along the maps between the target spaces. Having a map $f : (X', x') \rightarrow (X, x)$ of path-connected pointed spaces we can transform any HQFT (A, τ) with target X into an HQFT with target X' . It suffices to compose the characteristic maps with f and to apply (A, τ) . This induces a functor $f^* : Q_{d+1}(X, x) \rightarrow Q_{d+1}(X', x')$ (cf. Section 1.2).

2.2.1. Theorem. *Let $d \geq 1$ and X, X' be path-connected spaces with base points $x \in X, x' \in X'$. If a map $f : (X', x') \rightarrow (X, x)$ induces an isomorphism $\pi_i(X', x') \rightarrow \pi_i(X, x)$ for all $i \leq d$, then the functor $f^* : Q_{d+1}(X, x) \rightarrow Q_{d+1}(X', x')$ is an equivalence of categories.*

Proof. The standard obstruction theory shows that for any d -dimensional pointed manifold M , the composition with f defines a bijection $\text{Map}(M, X') \rightarrow \text{Map}(M, X)$. For a $(d+1)$ -dimensional cobordism (W, M_0, M_1) , the composition with f defines a bijection $\text{Map}(W, X') / \sim \rightarrow \text{Map}(W, X) / \sim$ where \sim is the equivalence relation introduced before the statement of Lemma 2.1.1. This lemma implies that from the viewpoint of HQFT's there is no difference between X -manifolds and X -cobordisms on one hand and X' -manifolds and X' -cobordisms on the other hand. In a formal language this means that $f^* : Q_{d+1}(X, x) \rightarrow Q_{d+1}(X', x')$ is an equivalence of categories.

2.2.2. Corollary. *A homotopy equivalence $(X', x') \rightarrow (X, x)$ of path-connected pointed spaces induces an equivalence of categories $Q_{d+1}(X, x) \rightarrow Q_{d+1}(X', x')$ for all d .*

2.2.3. Corollary. *For any connected CW-complex X with base point $x \in X$, the categories $Q_2(X, x)$ and $Q_2(K(\pi_1(X, x), 1))$ are equivalent.*

The equivalence is induced by the natural map $X \rightarrow K(\pi_1(X, x), 1)$ inducing the identity of the fundamental groups.

2.3. Independence of the base point. We show in this section that the notion of an HQFT with path-connected target X is essentially independent of the choice of a base point $x \in X$. To stress the role of the base point, we use here the terms (X, x) -manifolds and (X, x) -cobordisms for X -manifolds and X -cobordisms.

Let $\alpha : [0, 1] \rightarrow X$ be a path in X connecting the points $x = \alpha(0), y = \alpha(1)$. For every (X, x) -manifold M with characteristic map $g : M \rightarrow X$, consider a map $F_\alpha : M \times [0, 1] \rightarrow X$ such that

(*) $F_\alpha|_{M \times 0} = g$ and $F_\alpha(m \times t) = \alpha(t)$ for all the base points m of the components of M and all $t \in [0, 1]$.

The existence of F_α follows from the fact that $(M \times 0) \cup (\cup_m m \times [0, 1])$ is a strong deformation retract of $M \times [0, 1]$. Any two maps satisfying (*) are homotopic in the class of maps satisfying (*). Therefore, restricting F_α to the base $M \times 1$ we obtain a well-defined (X, y) -manifold. It is denoted M^α . The same construction applies to (X, x) -cobordisms and transforms an (X, x) -cobordism W into an (X, y) -cobordism W^α . Now, every HQFT (A, τ) with target (X, y) gives rise to an HQFT $({}^\alpha A, {}^\alpha \tau)$ with target (X, x) by $({}^\alpha A)_M = A_{M^\alpha}$ and $({}^\alpha \tau)(W) = \tau(W^\alpha)$. The action of homeomorphisms is defined by a similar formula. The notation is chosen so that $({}^{\alpha\beta} A, {}^{\alpha\beta} \tau) = ({}^\alpha({}^\beta A), {}^\alpha({}^\beta \tau))$ for paths $\alpha, \beta : [0, 1] \rightarrow X$ with $\alpha(1) = \beta(0)$. Note also that if two paths α, α' are homotopic rel $\{0, 1\}$, then $({}^\alpha A, {}^\alpha \tau) = ({}^{\alpha'} A, {}^{\alpha'} \tau)$. Thus, we can transport HQFT's along any path in the target space. This implies the claim at the beginning of this subsection.

These constructions may be applied to $y = x$. This gives a left action of $\pi_1(X, x)$ on HQFT's with target (X, x) . Observe that for any HQFT (A, τ) with target (X, x) and any $\alpha \in \pi_1(X, x)$, the HQFT $({}^\alpha A, {}^\alpha \tau)$ is isomorphic to (A, τ) . The isomorphism is given by the K -isomorphisms $\{\tau(M \times [0, 1], F_\alpha) : A_M \rightarrow ({}^\alpha A)_M\}_M$ where M runs over (X, x) -manifolds.

Part II. Crossed group-algebras and $(1+1)$ -dimensional HQFT's

3. Crossed group-algebras

3.1. Group-algebras. Let π be a group. A π -graded algebra or, briefly, a π -algebra over the ring K is an associative algebra L over K endowed with a splitting $L = \bigoplus_{\alpha \in \pi} L_\alpha$ such that each L_α is a projective K -module of finite type, $L_\alpha L_\beta \subset L_{\alpha\beta}$ for any $\alpha, \beta \in \pi$, and L has a (right and left) unit $1_L \in L_1$ where 1 is the neutral element of π .

An example of a π -algebra is provided by the group ring $L = K[\pi]$ with $L_\alpha = K\alpha$ for all $\alpha \in \pi$. More generally, for any associative unital K -algebra A we have a π -algebra $L = A[\pi]$ with $L_\alpha = A\alpha$ for $\alpha \in \pi$. Multiplication in $A[\pi]$ is given by $(a\alpha)(b\beta) = (ab)(\alpha\beta)$ where $a, b \in A$ and $\alpha, \beta \in \pi$.

We describe here a few simple operations on π -algebras. The direct sum $L \oplus L'$ of two π -algebras L, L' is a π -algebra defined by $(L \oplus L')_\alpha = L_\alpha \oplus L'_\alpha$ for $\alpha \in \pi$. The tensor product $L \otimes L'$ of π -algebras L, L' is a π -algebra defined by $(L \otimes L')_\alpha = L_\alpha \otimes L'_\alpha$. Multiplication in $L \oplus L'$ and $L \otimes L'$ is induced by multiplication in L, L' in the obvious way. The dual L^* of a π -algebra L is defined as the same module L with opposite multiplication $a \circ b = ba$ and the splitting $L^* = \bigoplus_{\alpha \in \pi} L_\alpha^*$ given by $L_\alpha^* = L_{\alpha^{-1}}$.

The group-algebras can be pulled back and pushed forward along group homomorphisms. Given a group homomorphism $q : \pi' \rightarrow \pi$ we can transform any π -algebra L into a π' -algebra $q^*(L)$ defined by $(q^*(L))_\alpha = L_{q(\alpha)}$ for any $\alpha \in \pi'$. Multiplication in $q^*(L)$ is induced by multiplication in L in the obvious way. If the kernel of q is finite then we can transform any π' -algebra L' into a π -algebra

$q_*(L')$. For $\alpha \in \pi$, set

$$(q_*(L'))_\alpha = \bigoplus_{u \in q^{-1}(\alpha)} L'_u.$$

Multiplication in $q_*(L')$ is induced by multiplication in L' . Note that $q_*(L') = L'$ as algebras:

$$q_*(L') = \bigoplus_{\alpha \in \pi} (q_*(L'))_\alpha = \bigoplus_{u \in \pi'} L'_u = L'.$$

A *Frobenius π -algebra* is a π -algebra L endowed with a symmetric K -bilinear form (inner product) $\eta : L \otimes L \rightarrow K$ such that

(3.1.1) $\eta(L_\alpha \otimes L_\beta) = 0$ if $\alpha\beta \neq 1$ and the restriction of η to $L_\alpha \otimes L_{\alpha^{-1}}$ is non-degenerate for all $\alpha \in \pi$;

(3.1.2) $\eta(ab, c) = \eta(a, bc)$ for any $a, b, c \in L$.

Recall that for K -modules P, Q , a bilinear form $P \otimes Q \rightarrow K$ is *non-degenerate* if both adjoint homomorphisms $P \rightarrow Q^* = \text{Hom}(Q, K)$ and $Q \rightarrow P^*$ are isomorphisms. The group ring $L = K[\pi]$ is a Frobenius π -algebra with inner product determined by $\eta(\alpha, \beta) = 1$ if $\alpha\beta = 1$ and $\eta(\alpha, \beta) = 0$ if $\alpha\beta \neq 1$ where $\alpha, \beta \in \pi$.

It is clear that the direct sum and the tensor product of Frobenius π -algebras L, L' are Frobenius π -algebras; the inner products in L, L' extend to $L \oplus L'$ (resp. to $L \otimes L'$) by linearity (resp. by multiplicativity). The inner product in L induces an inner product in the dual π -algebra L^* via the equality of modules $L^* = L$; this makes L^* a Frobenius π -algebra. The pull-backs and push-forwards of Frobenius group-algebras are Frobenius group-algebras in a natural way.

3.2. Crossed π -algebras. A *crossed π -algebra* over K is a Frobenius π -algebra over K endowed with a group homomorphism $\varphi : \pi \rightarrow \text{Aut}(L)$ satisfying the following four axioms:

(3.2.1) for all $\beta \in \pi$, $\varphi_\beta = \varphi(\beta)$ is an algebra automorphism of L preserving η and such that $\varphi_\beta(L_\alpha) \subset L_{\beta\alpha\beta^{-1}}$ for all $\alpha \in \pi$;

(3.2.2) $\varphi_\beta|_{L_\beta} = \text{id}$, for all $\beta \in \pi$;

(3.2.3) for any $a \in L_\alpha, b \in L_\beta$, we have $\varphi_\beta(a)b = ba$;

(3.2.4) for any $\alpha, \beta \in \pi$ and any $c \in L_{\alpha\beta\alpha^{-1}\beta^{-1}}$ we have

$$(3.2.a) \quad \text{Tr}(c\varphi_\beta : L_\alpha \rightarrow L_\alpha) = \text{Tr}(\varphi_{\alpha^{-1}}c : L_\beta \rightarrow L_\beta).$$

Here Tr is the K -valued trace of endomorphisms of projective K -modules of finite type, see for instance [Tu, Appendix I]. If K is a field then Tr is the standard trace of matrices. The homomorphism on the left-hand side of (3.2.a) sends any $a \in L_\alpha$ into $c\varphi_\beta(a) \in L_\alpha$ and the homomorphism on the right-hand side sends any $b \in L_\beta$ into $\varphi_{\alpha^{-1}}(cb) \in L_\beta$.

Note a few corollaries of the definition. The module $L_1 \subset L$ is a commutative associative K -algebra with unit. (The commutativity follows from (3.2.3) since $\varphi_1 = \text{id}$.) The restriction of η to L_1 is non-degenerate so that the pair (L_1, η) is a commutative Frobenius algebra over K . The group π acts on L_1 by algebra automorphisms preserving η .

The algebra L_1 acts on each L_α by multiplications on the left and on the right. Axiom (3.2.3) with $\beta = 1$ implies that left and right multiplications by elements of L_1 coincide.

Axiom (3.2.4) with $\beta = 1$ and $c = 1_L \in L_1$ implies that for any $\alpha \in \pi$,

$$\text{Dim } L_\alpha = \text{Tr}(\text{id} : L_\alpha \rightarrow L_\alpha) = \text{Tr}(\varphi_1 : L_\alpha \rightarrow L_\alpha) = \text{Tr}(\varphi_{\alpha^{-1}} : L_1 \rightarrow L_1) \in K.$$

Note that if K is a field then $\text{Dim } L_\alpha = (\dim L_\alpha)1_K$ where \dim is the standard integer-valued dimension of vector spaces over K and $1_K \in K$ is the unit of K . In particular if K is a field of characteristic 0 then the dimensions of all $\{L_\alpha\}$ are determined by the character of the representation $\varphi|_{L_1} : \pi \rightarrow \text{Aut}(L_1)$.

We define a category, $Q_2(\pi) = Q_2(\pi; K)$, whose objects are crossed π -algebras over K . A morphism $L \rightarrow L'$ in this category is an algebra homomorphism $L \rightarrow L'$ mapping each L_α to L'_α , preserving the unit and the inner product and commuting with the action of π . It is easy to show (we shall not use it) that all morphisms in the category $Q_2(\pi)$ are isomorphisms.

The operations on group-algebras discussed in Section 3.1 apply also to crossed group-algebras. The direct sum and the tensor product of crossed π -algebras L, L' are crossed π -algebras: the action of π on L, L' extends to $L \oplus L'$ (resp. to $L \otimes L'$) by linearity (resp. by multiplicativity). The action of π on L induces an action of π on the dual π -algebra L^* via the equality $L^* = L$; this makes L^* a crossed π -algebra.

Given a group homomorphism $q : \pi' \rightarrow \pi$ and a crossed π -algebra L we can provide the π' -algebra $L' = q^*(L)$ with the structure of a crossed π' -algebra. The inner product in L induces an inner product in L' in the obvious way. The action of π on L induces an action of π' on L' by

$$\varphi_\beta = \varphi_{q(\beta)} : L'_\alpha = L_{q(\alpha)} \rightarrow L_{q(\beta\alpha\beta^{-1})} = L'_{\beta\alpha\beta^{-1}}$$

for $\alpha, \beta \in \pi'$. A similar push-forward construction for crossed group-algebras will be discussed in Section 10.3.

A useful operation on a crossed π -algebra (L, η, φ) consists in rescaling the inner product: for $k \in K^*$, the triple $(L, k\eta, \varphi)$ is also a crossed π -algebra.

Crossed algebras over the trivial group $\pi = 1$ are nothing but commutative Frobenius algebras over K whose underlying K -modules are projective of finite type. Each such Frobenius algebra A determines a crossed π -algebra $A[\pi]$ over any group π : it suffices to take the pull-back of A along the trivial homomorphism $\pi \rightarrow \{1\}$. Clearly, $A[\pi]_\alpha = A\alpha$ for all $\alpha \in \pi$. The underlying π -algebra of $A[\pi]$ is the one described at the beginning of Section 3.1. The group π acts on $A[\pi]$ by permutations of the copies of A .

In the remaining part of Section 3 we describe several constructions of crossed group-algebras. Our principal result is a classification of semisimple crossed group-algebras in terms of 2-dimensional group cohomology.

3.3. Example: crossed π -algebras from 2-cocycles. Let $\{\theta_{\alpha,\beta} \in K^*\}_{\alpha,\beta \in \pi}$ be a normalized 2-cocycle of the group π with values in the multiplicative group

K^* . Thus

$$(3.3.a) \quad \theta_{\alpha,\beta} \theta_{\alpha\beta,\gamma} = \theta_{\alpha,\beta\gamma} \theta_{\beta,\gamma}$$

for any $\alpha, \beta, \gamma \in \pi$ and $\theta_{1,1} = 1$. We define a crossed π -algebra $L = L^\theta$ as follows.

For $\alpha \in \pi$, let L_α be the free K -module of rank one generated by a vector l_α , i.e., $L_\alpha = Kl_\alpha$. Multiplication is defined by $l_\alpha l_\beta = \theta_{\alpha,\beta} l_{\alpha\beta}$. Note that multiplication induces an isomorphism $L_\alpha \otimes_K L_\beta \rightarrow L_{\alpha\beta}$. The inner product η on L is determined by $\eta(l_\alpha, l_{\alpha^{-1}}) = \theta_{\alpha,\alpha^{-1}}$ for all α and $\eta(l_\alpha, l_\beta) = 0$ for $\beta \neq \alpha^{-1}$. The value of the endomorphism $\varphi_\beta : L \rightarrow L$ on each l_α is uniquely determined by the condition $\varphi_\beta(l_\alpha) l_\beta = l_\beta l_\alpha$.

Let us verify the axioms of a crossed π -algebra. The associativity of multiplication follows from (3.3.a). Substituting $\beta = \gamma = 1$ in (3.3.a) we obtain that $\theta_{\alpha,1} = 1$ for all $\alpha \in \pi$. Substituting $\alpha = \beta = 1$ in (3.3.a), we obtain that $\theta_{1,\gamma} = 1$ for all $\gamma \in \pi$. This implies that $1_L = l_1 \in L_1$ is both right and left unit of L .

Substituting $\beta = \alpha^{-1}$ and $\gamma = \alpha$ in (3.3.a) we obtain that $\theta_{\alpha,\alpha^{-1}} = \theta_{\alpha^{-1},\alpha}$ for all $\alpha \in \pi$. Hence the form η is symmetric. Axiom (3.1.1) follows from definitions. A direct computation shows that $\eta(l_\alpha, l_\beta) = \eta(l_\alpha l_\beta, 1_L)$ for all α, β . Therefore $\eta(a, b) = \eta(ab, 1_L)$ for all $a, b \in L$. This implies (3.1.2).

Let us verify (3.2.1). For $\alpha, \alpha', \beta \in \pi$, we have

$$\begin{aligned} \varphi_\beta(l_\alpha l_{\alpha'}) l_\beta &= \theta_{\alpha,\alpha'} \varphi_\beta(l_{\alpha\alpha'}) l_\beta = \theta_{\alpha,\alpha'} l_\beta l_{\alpha\alpha'} = l_\beta l_\alpha l_{\alpha'} \\ &= \varphi_\beta(l_\alpha) l_\beta l_{\alpha'} = \varphi_\beta(l_\alpha) \varphi_\beta(l_{\alpha'}) l_\beta. \end{aligned}$$

This implies that $\varphi_\beta(l_\alpha l_{\alpha'}) = \varphi_\beta(l_\alpha) \varphi_\beta(l_{\alpha'})$. Substituting $\alpha = \alpha' = 1$, we obtain that $\varphi_\beta(1_L) = 1_L$ and therefore $\varphi_\beta|_{L_1} = \text{id}$ for all $\beta \in \pi$. For $a, b \in L$, we have

$$\eta(\varphi_\beta(a), \varphi_\beta(b)) = \eta(\varphi_\beta(a) \varphi_\beta(b), 1_L) = \eta(\varphi_\beta(ab), 1_L).$$

Note that $\oplus_{\alpha \neq 1} L_\alpha$ is orthogonal to L_1 and $\varphi_\beta|_{L_1} = \text{id}$. Therefore $\eta(\varphi_\beta(ab), 1_L) = \eta(ab, 1_L) = \eta(a, b)$ which proves the invariance of η under φ_β .

Axioms (3.2.2) and (3.2.3) follow directly from the definition of φ_β .

Let us check the last axiom (3.2.4). The homomorphism $c\varphi_\beta : L_\alpha \rightarrow L_\alpha$ sends l_α into kl_α with certain $k \in K^*$. The homomorphism $\varphi_{\alpha^{-1}}c : L_\beta \rightarrow L_\beta$ sends l_β into $k'l_\beta$ with $k' \in K^*$. Note that

$$kl_\alpha l_\beta = c\varphi_\beta(l_\alpha) l_\beta = cl_\beta l_\alpha = l_\alpha \varphi_{\alpha^{-1}}(cl_\beta) = k'l_\alpha l_\beta.$$

Therefore $k = k'$ and

$$\text{Tr}(c\varphi_\beta : L_\alpha \rightarrow L_\alpha) = k = k' = \text{Tr}(\varphi_{\alpha^{-1}}c : L_\beta \rightarrow L_\beta).$$

It is easy to see that the isomorphism class of the crossed π -algebra L^θ depends only on the cohomology class $\theta \in H^2(\pi; K^*)$ represented by the 2-cocycle $\{\theta_{\alpha,\beta}\}_{\alpha,\beta \in \pi}$. It is obvious that $L^{\theta\theta'} = L^\theta \otimes L^{\theta'}$ for any $\theta, \theta' \in H^2(\pi; K^*)$ (we

use multiplicative notation for the group operation in $H^2(\pi; K^*)$). The crossed π -algebra corresponding to the neutral element of $H^2(\pi; K^*)$ coincides with the crossed π -algebra $K[\pi]$ defined at the end of Section 3.2 where we take $A = K$ and set $\eta(a, b) = ab$ for $a, b \in K$.

3.4. Transfer for crossed algebras. Let $G \subset \pi$ be a subgroup of π of finite index $n = [\pi : G]$. We show that each crossed G -algebra (L, η, φ) gives rise to a crossed π -algebra $(\tilde{L}, \tilde{\eta}, \tilde{\varphi})$. It is called the *transfer* of L .

For each right coset class $i \in G \backslash \pi$, fix a representative $\omega_i \in \pi$ so that $i = G\omega_i \subset \pi$. For $\alpha \in \pi$, set

$$N(\alpha) = \{i \in G \backslash \pi \mid \omega_i \alpha \omega_i^{-1} \in G\}$$

and

$$\tilde{L}_\alpha = \bigoplus_{i \in N(\alpha)} L_{\omega_i \alpha \omega_i^{-1}}.$$

(In particular, if α is not conjugated to an element of G , then $\tilde{L}_\alpha = 0$.) We provide $\tilde{L} = \bigoplus_\alpha \tilde{L}_\alpha$ with multiplication as follows. The multiplication $\tilde{L}_\alpha \otimes \tilde{L}_\beta \rightarrow \tilde{L}_{\alpha\beta}$ with $\alpha, \beta \in \pi$ sends $L_{\omega_i \alpha \omega_i^{-1}} \otimes L_{\omega_j \beta \omega_j^{-1}}$ into 0 if $i \neq j$ and is induced by multiplication in L

$$L_{\omega_i \alpha \omega_i^{-1}} \otimes L_{\omega_i \beta \omega_i^{-1}} \rightarrow L_{\omega_i \alpha \beta \omega_i^{-1}}$$

if $i = j \in N(\alpha) \cap N(\beta)$. Clearly, \tilde{L} is an associative algebra.

By definition, $\tilde{L}_1 = \bigoplus_{i \in G \backslash \pi} L_1$ is a direct sum of n copies of L_1 . The corresponding sum of n copies of $1_L \in L_1$ is the unit $1_{\tilde{L}} \in \tilde{L}_1$.

The inner product $\tilde{\eta} : \tilde{L} \otimes \tilde{L} \rightarrow K$ is determined by

$$\tilde{\eta}|_{\tilde{L}_\alpha \otimes \tilde{L}_{\alpha^{-1}}} = \bigoplus_{i \in N(\alpha) = N(\alpha^{-1})} \eta|_{L_{\omega_i \alpha \omega_i^{-1}} \otimes L_{\omega_i \alpha^{-1} \omega_i^{-1}}}$$

where α runs over π . Clearly, $\tilde{\eta}$ is a symmetric bilinear form verifying (3.1.1). It is easy to deduce from definitions that $\tilde{\eta}(a, b) = \tilde{\eta}(ab, 1_{\tilde{L}})$ for any $a, b \in \tilde{L}$. This implies (3.1.2) for $\tilde{\eta}$.

The action $\tilde{\varphi}$ of π on \tilde{L} is defined as follows. The group π acts on $G \backslash \pi$ by $\beta(i) = i\beta^{-1}$ for $\beta \in \pi, i \in G \backslash \pi$. Then $G\omega_{\beta(i)} = G\omega_i\beta^{-1}$ so that $\beta_i = \omega_{\beta(i)}\beta\omega_i^{-1} \in G$. For any given $\alpha \in \pi$, the map $i \mapsto \beta(i)$ sends bijectively $N(\alpha)$ onto $N(\beta\alpha\beta^{-1})$. For every $i \in N(\alpha)$, we have the homomorphism

$$(3.4.a) \quad \varphi_{\beta_i} : L_{\omega_i \alpha \omega_i^{-1}} \rightarrow L_{\omega_{\beta(i)} \beta \alpha \beta^{-1} (\omega_{\beta(i)})^{-1}}.$$

The direct sum of these homomorphisms over all $i \in N(\alpha)$ is a homomorphism $\tilde{\varphi}_\beta : \tilde{L}_\alpha \rightarrow \tilde{L}_{\beta\alpha\beta^{-1}}$. It extends by additivity to an endomorphism, $\tilde{\varphi}_\beta$, of \tilde{L} . An easy computation shows that $(\beta\beta')_i = \beta_{\beta'(i)}\beta'_i$ for any $\beta, \beta' \in \pi$. This implies that $\tilde{\varphi}_{\beta\beta'} = \tilde{\varphi}_\beta \tilde{\varphi}_{\beta'}$. It follows from definitions that $\tilde{\varphi}_1 = \text{id}$. Hence $\tilde{\varphi}$ is an action of π on \tilde{L} .

Note that the homomorphism (3.4.a) preserves multiplication and inner product in L and therefore $\tilde{\varphi}_\beta$ preserves multiplication and inner product in \tilde{L} . This implies (3.2.1).

It is clear that for any $i \in N(\beta)$, we have $\beta(i) = i$ and $\beta_i = \omega_i \beta \omega_i^{-1}$. Therefore, by (3.2.2), the homomorphism (3.4.a) is the identity in the case $\alpha = \beta$. This implies (3.2.2) for the action $\tilde{\varphi}_\beta$.

Let us verify axiom (3.2.3) for \tilde{L} . Let $a \in L_{\omega_i \alpha \omega_i^{-1}} \subset \tilde{L}_\alpha, b \in L_{\omega_j \beta \omega_j^{-1}} \subset \tilde{L}_\beta$ with $i \in N(\alpha), j \in N(\beta)$. By definition,

$$\tilde{\varphi}_\beta(a) = \varphi_{\beta_i}(a) \in L_{\omega_{\beta(i)} \beta \alpha \beta^{-1} (\omega_{\beta(i)})^{-1}}.$$

The inclusion $j \in N(\beta)$ implies that $\beta(j) = j$. Therefore, if $i \neq j$ then $\beta(i) \neq \beta(j) = j$. Hence in the case $i \neq j$, we have $\tilde{\varphi}_\beta(a)b = 0 = ba$. Assume that $i = j$. Then $\beta(i) = \beta(j) = j$ and $\beta_i = \omega_i \beta \omega_i^{-1}$. By axioms (3.2.3) and (3.2.2) for L ,

$$\tilde{\varphi}_\beta(a)b = \varphi_{\beta_i}(a)b = \varphi_{\omega_i \beta \omega_i^{-1}}(a)b = \varphi_{\omega_j \beta \omega_j^{-1}}(a)b = ba.$$

Let us verify that for any $\alpha, \beta \in \pi$ and any $c \in \tilde{L}_{\alpha \beta \alpha^{-1} \beta^{-1}}$ we have

$$(3.4.b) \quad \text{Tr}(c \tilde{\varphi}_\beta : \tilde{L}_\alpha \rightarrow \tilde{L}_\alpha) = \text{Tr}(\tilde{\varphi}_{\alpha^{-1}} c : \tilde{L}_\beta \rightarrow \tilde{L}_\beta).$$

Since both sides of this formula are linear with respect to c , it suffices to consider the case where

$$c \in L_{\omega_i \alpha \beta \alpha^{-1} \beta^{-1} \omega_i^{-1}} \subset \tilde{L}_{\alpha \beta \alpha^{-1} \beta^{-1}}$$

with $i \in N(\alpha \beta \alpha^{-1} \beta^{-1})$. A direct application of the definitions shows that both sides of (3.4.b) are equal to 0 unless $i \in N(\alpha) \cap N(\beta)$. If $i \in N(\alpha) \cap N(\beta)$ then the left-hand side of (3.4.b) equals the trace of the endomorphism $c \varphi_{\omega_i \beta \omega_i^{-1}}$ of $\tilde{L}_{\omega_i \alpha \omega_i^{-1}}$ and the right-hand side of (3.4.b) equals the trace of the endomorphism $\varphi_{\omega_i \alpha^{-1} \omega_i^{-1}} c$ of $\tilde{L}_{\omega_i \beta \omega_i^{-1}}$. The equality of these two traces follows from axiom (3.2.4) for L .

We leave it as an exercise to the reader to verify that the isomorphism class of the crossed π -algebra \tilde{L} does not depend on the choice of the representatives $\{\omega_i\}$.

3.5. Semisimple crossed π -algebras. A crossed π -algebra $L = \bigoplus_{\alpha \in \pi} L_\alpha$ is said to be *semisimple* if the commutative K -algebra L_1 is semisimple, i.e., if L_1 is a direct sum of several copies of the ring K . Note that direct sums, tensor products, pull-backs, duals, and transfers of semisimple crossed algebras are semisimple. In this subsection we study the structure of semisimple crossed π -algebras.

We first briefly discuss semisimple commutative algebras of finite type over K . Each such algebra, R , is a direct sum of a finite number of copies of K , say $\{K_u\}_u$. Let $i_u \in K_u$ be the unit element of K_u . We have $1_R = \sum_u i_u$ and $i_u i_v = \delta_u^v i_u$ for any u, v where δ is the Kronecker symbol. Each element $r \in R$ can be uniquely written in the form $r = \sum_u r_u i_u$ with $r_u \in K$. It is clear that r

is an idempotent (i.e., $r^2 = r$) if and only if $r_u \in \{0, 1\}$ for all u . We call the set $\{u \mid r_u \neq 0\}$ the support of r . Now, the product of two idempotents is zero if and only if their supports are disjoint. This characterizes the set $\{i_u\}_u$ as the unique basis of R consisting of mutually annihilating idempotents. We denote this set by $\text{bas}(R)$; its elements are called *basic idempotents* of R . In particular, any algebra automorphism of R acts by permutations on $\text{bas}(R)$.

Let us come back to crossed algebras. By a *basic idempotent* of a semisimple crossed π -algebra L we shall mean a basic idempotent of L_1 . Set $\text{bas}(L) = \text{bas}(L_1) \subset L_1$. The equalities $ii' = \delta_{i,i'}^i$ for $i, i' \in \text{bas}(L)$ and $1_L = \sum_{i \in \text{bas}(L)} i$ imply that L is a direct sum of its subalgebras iL with $i \in \text{bas}(L)$, i.e., $L = \bigoplus_{i \in \text{bas}(L)} iL$ where

$$(3.5.a) \quad iL = \bigoplus_{\alpha \in \pi} iL_\alpha.$$

The subalgebras $\{iL\}_{i \in \text{bas}(L)}$ of L are mutually annihilating and orthogonal with respect to the inner product. The action φ of π permutes these subalgebras.

We say that L is *simple* if it is semisimple and the action of π on L_1 is transitive on $\text{bas}(L)$. Observe that every semisimple crossed π -algebra L splits uniquely as a direct sum of simple crossed π -algebras. Indeed, the algebra L_1 splits as a direct sum of its subalgebras $\{mL_1\}_m$ generated by the orbits $m \subset \text{bas}(L)$ of the π -action on $\text{bas}(L)$. Then L is a direct (orthogonal) sum of the subalgebras $mL = \bigoplus_{i \in m} iL$ preserved by the action of π . Clearly, these subalgebras are simple crossed π -algebras.

For a simple crossed π -algebra L , any two elements of $\text{bas}(L)$ can be obtained from each other by the action of π . Therefore the value $\eta(i, i) \in K$ with $i \in \text{bas}(L)$ does not depend on the choice of i . The non-degeneracy of η implies that $\eta(i, i) \in K^*$. We call L *normalized* if $\eta(i, i) = 1$ for any $i \in \text{bas}(L)$. It is clear that any simple crossed π -algebra is obtained from a normalized simple crossed π -algebra by rescaling, see Section 3.2.

3.5.1. Examples. 1. The crossed π -algebra L associated with a 2-dimensional cohomology class of π as in Section 3.3 is simple and normalized since $L_1 = K1_L$ and $\eta(1_L, 1_L) = 1$.

2. Let G be a subgroup of π of finite index n and L be a crossed G -algebra associated with $\theta \in H^2(G; K^*)$. Then the crossed π -algebra $\tilde{L} = L^{\pi, G, \theta}$ obtained as the transfer of L is semisimple because the algebra \tilde{L}_1 is a direct sum of n copies of $L_1 = K$. The action of π on $\tilde{L}_1 = \bigoplus_{i \in G \setminus \pi} L_1$ permutes the copies of L_1 via the natural action of π on $G \setminus \pi$. Hence the action of π on $\text{bas}(\tilde{L}) = G \setminus \pi$ is transitive and the crossed π -algebra \tilde{L} is simple. It is easy to check that \tilde{L} is normalized. We have a distinguished element $i_0(\theta) \in \text{bas}(\tilde{L})$: this is the unit element of the copy of L_1 corresponding to $G \setminus G \in G \setminus \pi$.

3. The crossed π -algebra $A[\pi]$ considered at the end of Section 3.2 is semisimple if and only if A is semisimple.

The following theorem yields a classification of simple crossed π -algebras over

fields of characteristic 0. We introduce the following notation. Denote by $C(\pi)$ the set of pairs (a subgroup $G \subset \pi$ of finite index, a cohomology class $\theta \in H^2(G; K^*)$). The group π acts on $C(\pi)$ as follows: $\alpha(G, \theta) = (\alpha G \alpha^{-1}, \alpha_*(\theta))$ where $\alpha \in \pi$ and $\alpha_* : H^2(G; K^*) \rightarrow H^2(\alpha G \alpha^{-1}; K^*)$ is the isomorphism induced by the conjugation by α . Denote by $D(\pi)$ the set of isomorphism classes of pairs (a normalized simple crossed π -algebra (L, φ, η) , a basic idempotent $i_0 \in L_1$). The group π acts on $D(\pi)$ by $\alpha(L, i_0) = (L, \varphi_\alpha(i_0))$ where $\alpha \in \pi$. The set of orbits $D(\pi)/\pi$ is just the set of isomorphism classes of normalized simple crossed π -algebras.

3.6. Theorem. *Let the ground ring K be a field of characteristic 0. Then the formula $(G, \theta) \mapsto (L^{\pi, G, \theta}, i_0(\theta))$ defines a π -equivariant bijection $C(\pi) \rightarrow D(\pi)$. Hence*

$$D(\pi)/\pi = C(\pi)/\pi.$$

Proof. We define the inverse mapping $D(\pi) \rightarrow C(\pi)$ as follows. Let (L, φ, η) be a normalized simple crossed π -algebra with distinguished basic idempotent $i_0 \in L_1$. We compute the dimension of $iL_\alpha \subset L_\alpha$ for $i \in \text{bas}(L), \alpha \in \pi$ by applying (3.2.4) to $\beta = 1$ and $c = i \in L_1$. This gives

$$\begin{aligned} (3.6.a) \quad \dim(iL_\alpha) &= \text{Tr}(a \mapsto ia : L_\alpha \rightarrow L_\alpha) \\ &= \text{Tr}(a \mapsto \varphi_{\alpha^{-1}}(ia) : L_1 \rightarrow L_1) = \begin{cases} 1, & \text{if } \varphi_\alpha(i) = i, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Let $G = \{\alpha \in \pi \mid \varphi_\alpha(i_0) = i_0\}$ be the stabilizer of i_0 with respect to the action of π on $\text{bas}(L)$. We have $\dim(i_0 L_\alpha) = 1$ for all $\alpha \in G$. For any $\alpha \in G \setminus \{1\}$, choose a generator $s_\alpha \in i_0 L_\alpha$. For $\alpha = 1$ set $s_\alpha = i_0 \in i_0 L_1$. For any $\alpha, \beta \in G$, we have $s_\alpha s_\beta = \theta_{\alpha, \beta} s_{\alpha\beta}$ with $\theta_{\alpha, \beta} \in K$. We claim that $\theta_{\alpha, \beta} \in K^*$. Indeed, by the non-degeneracy of η we have $\eta(s_\beta s_{\beta^{-1}}, 1_L) = \eta(s_\beta, s_{\beta^{-1}}) \in K^*$. Therefore $s_\beta s_{\beta^{-1}} = k i_0$ with $k \in K^*$ and

$$\theta_{\alpha, \beta} \theta_{\alpha\beta, \beta^{-1}} s_\alpha = \theta_{\alpha, \beta} s_{\alpha\beta} s_{\beta^{-1}} = s_\alpha s_\beta s_{\beta^{-1}} = k i_0 s_\alpha = k s_\alpha.$$

Hence, $\theta_{\alpha, \beta} \in K^*$. The associativity of multiplication in L and the choice $s_1 = i_0$ imply that $\{\theta_{\alpha, \beta}\}_{\alpha, \beta}$ is a normalized 2-cocycle of G representing a certain cohomology class $\theta \in H^2(G; K^*)$. Under a different choice of the generators $\{s_\alpha\}_{\alpha \in G}$ we obtain a cohomological 2-cocycle. Thus, the formula $(L, i_0) \mapsto (G, \theta)$ yields a well defined mapping $D(\pi) \rightarrow C(\pi)$. It follows from definitions that this mapping is π -equivariant. (The key point is that the action of π on L via φ preserves multiplication.)

We can apply the construction of the previous paragraph to the crossed π -algebra \tilde{L} derived as in Sections 3.3, 3.4 from a subgroup of finite index $G \subset \pi$ and a cohomology class $\theta \in H^2(G; K^*)$. The distinguished basic idempotent of \tilde{L} is $i_0 = G \setminus G \in G \setminus \pi = \text{bas}(\tilde{L})$. The stabilizer of i_0 with respect to the natural action

of π is G . As the representative $\omega_{i_0} \in G$ of i_0 (used in Section 3.3) we take $1 \in G$. For $\alpha \in G$, we take as the generator s_α of $i_0 \tilde{L}_\alpha = L_\alpha$ the element l_α used in Section 3.3. Now, it is obvious that the construction of the previous paragraph applied to the pair (\tilde{L}, i_0) gives (G, θ) . Thus, the mapping $D(\pi) \rightarrow C(\pi)$ constructed above is the left inverse of the mapping $C(\pi) \rightarrow D(\pi)$ in the statement of the theorem.

To accomplish the proof, it suffices to show that the mapping $D(\pi) \rightarrow C(\pi)$ is injective. We need to prove that any normalized simple crossed π -algebra (L, φ, η) with distinguished basic idempotent $i_0 \in L_1$ can be uniquely reconstructed from its subalgebra $i_0 L = \bigoplus_{\alpha \in G} K s_\alpha$ where $G \subset \pi$ is the stabilizer of i_0 and s_α is a generator of $i_0 L_\alpha$. Note first that the form η on L is completely determined by the formulas $\eta(a, b) = \eta(ab, 1_L)$ and $\eta(\sum_{i \in \text{bas}(L)} k_i i, 1_L) = \sum_{i \in \text{bas}(L)} k_i$ for any $k_i \in K$. Since the action of π on $\text{bas}(L)$ is transitive, for each $i \in \text{bas}(L)$ there is an element $\omega_i \in \pi$ such that $\varphi_{\omega_i}(i_0) = i$. We take $\omega_{i_0} = 1$. The homomorphism $\varphi_{\omega_i} : L \rightarrow L$ maps $i_0 L$ isomorphically onto iL . Therefore the elements $\varphi_{\omega_i}(s_\alpha)$ with $i \in \text{bas}(L)$ and $\alpha \in G$ form an additive basis of L . The product of two basis elements is computed by

$$\varphi_{\omega_i}(s_\alpha) \varphi_{\omega_j}(s_\beta) = \begin{cases} \varphi_{\omega_i}(s_\alpha s_\beta), & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

It remains to recover the action φ of π on L . For $\alpha \in G$, the homomorphism $\varphi_\alpha : i_0 L \rightarrow i_0 L$ is uniquely determined by the condition $\varphi_\alpha(s_\beta) s_\alpha = s_\alpha s_\beta$ for all $\beta \in G$. Each $\beta \in \pi$ splits as a product $\omega_i \alpha$ with $\alpha \in G$. This gives $\varphi_\beta = \varphi_{\omega_i} \varphi_\alpha$ and computes the restriction of φ_β to $i_0 L$. Knowing these restrictions for all $\beta \in \pi$, we can uniquely recover the whole action of π on L because each basis element of L given above has the form $\varphi_\alpha(s)$ with $s \in i_0 L$.

3.7. Corollary. *Let K be a field of characteristic 0. Then there is a bijection from $H^2(\pi; K^*)$ onto the set of isomorphism classes of crossed π -algebras L such that $\dim L_1 = 1$ and $\eta(1_L, 1_L) = 1_K$.*

4. Two-dimensional HQFT's

We shall relate $(1+1)$ -dimensional HQFT's to crossed π -algebras where π is the fundamental group of the target space. In view of Corollary 2.2.3, we can restrict ourselves to HQFT's with target $K(\pi, 1)$.

4.1. Theorem. *Let π be a group. Every $(1+1)$ -dimensional HQFT with target $K(\pi, 1)$ determines an “underlying” crossed π -algebra. This establishes an equivalence between the category $Q_2(K(\pi, 1))$ of $(1+1)$ -dimensional HQFT's with target $K(\pi, 1)$ and the category $Q_2(\pi)$ of crossed π -algebras. The direct sums, tensor products, pull-backs, duals and transfers for $(1+1)$ -dimensional HQFT's correspond to direct sums, tensor products, pull-backs, duals and transfers for crossed algebras.*

It is understood that the pull-backs of HQFT's are effected along maps between Eilenberg-MacLane spaces of type $K(\pi, 1)$. By transfers of HQFT's we mean the transfers described in Section 1.6.

4.2. Corollary. *There is a bijective correspondence between the isomorphism classes of $(1 + 1)$ -dimensional HQFT's with target $K(\pi, 1)$ and the isomorphism classes of crossed π -algebras.*

Note also another corollary of Theorem 4.1: the group of endomorphisms of a $(1 + 1)$ -dimensional HQFT with target $K(\pi, 1)$ is isomorphic to the group of endomorphisms of the underlying crossed π -algebra.

We complement Theorem 4.1 with a description of crossed π -algebras corresponding to semi-cohomological HQFT's (cf. Section 1.6).

4.3. Theorem. *The underlying crossed π -algebra of a cohomological (resp. semi-cohomological) $(1 + 1)$ -dimensional HQFT with target $K(\pi, 1)$ is normalized and simple (resp. semisimple). The converse is also true provided the ground ring K is a field of characteristic 0.*

Since the splitting of a semisimple crossed group-algebra as a direct sum of simple crossed group-algebras is unique (up to permutation of summands), we obtain the following corollary.

4.4. Corollary. *Let the ground ring K be a field of characteristic 0. The splitting of a semi-cohomological $(1 + 1)$ -dimensional HQFT over K into a direct sum of rescaled cohomological HQFT's is unique.*

In the remaining part of Section 4 we construct the underlying crossed algebra of a $(1 + 1)$ -dimensional HQFT and show the functoriality of this construction. The proof of Theorems 4.1 and 4.3 will be given in Section 5.

Throughout this section we fix a group π and a $(1 + 1)$ -dimensional HQFT (A, τ) with target $X = K(\pi, 1)$ and base point $x \in X$.

4.5. Preliminaries on $(1 + 1)$ -dimensional HQFT's. Here we reformulate the data provided by a $(1 + 1)$ -dimensional HQFT (A, τ) in a form convenient for the sequel. Observe first that a 1-dimensional connected X -manifold M is just a pointed oriented circle endowed with a map into X sending the base point into x . This is nothing but a loop in X with endpoints in x . The K -module A_M depends only on the class of this loop in $\pi_1(X, x) = \pi$, see Section 2.1. In this way, for each $\alpha \in \pi$ we obtain a K -module denoted L_α . A non-connected X -manifold M is a finite non-ordered family $\{\alpha_i\}_i$ of loops in X and $A_M = \otimes_i L_{\alpha_i}$. If $M = \emptyset$, then $A_M = K$.

Let W be a compact oriented surface with pointed oriented boundary endowed with a map $g : W \rightarrow X$ sending the base points of all the components of ∂W into x . We write C_+ for a component $C \subset \partial W$ with orientation induced from

W and we write C_- for C with opposite orientation. Let $\{(C_-^p, \alpha_p \in \pi)\}_p$ be the components of ∂W whose orientations are opposite to the one induced from W . Here α_p is represented by the restriction of g to C_-^p . Let $\{(C_+^q, \beta_q \in \pi)\}_q$ be the components of ∂W whose orientations are induced from W . Here β_q is represented by the restriction of g to C_+^q . We view W as an X -cobordism between $\cup_p (C_-^p, \alpha_p)$ and $\cup_q (C_+^q, \beta_q)$. By the remarks of Section 2.1, the HQFT (A, τ) gives rise to a homomorphism

$$\tau(W) \in \text{Hom}_K\left(\bigotimes_p L_{\alpha_p}, \bigotimes_q L_{\beta_q}\right) = \left(\bigotimes_p L_{\alpha_p}^*\right) \otimes \left(\bigotimes_q L_{\beta_q}\right).$$

This homomorphism is preserved under any homotopy of the map $g : W \rightarrow X$ relative to the base points on ∂W . The axioms (1.2.6) and (1.2.7) tell us that the homomorphism $\tau(W)$ is multiplicative under the gluing of cobordisms and that $\tau(W) = \text{id}$ if W is a cylinder as in (1.2.7).

4.6. Annuli and discs with holes. In this subsection we discuss the structures of X -cobordisms on annuli and discs with 2 holes.

Let C denote the annulus $S^1 \times [0, 1]$. We fix an orientation of C once for all. Set $C^0 = S^1 \times 0 \subset \partial C$ and $C^1 = S^1 \times 1 \subset \partial C$. Let us provide C^0, C^1 with base points $c^0 = s \times 0, c^1 = s \times 1$, respectively, where $s \in S^1$. For any signs $\varepsilon, \mu = \pm$ we denote by $C_{\varepsilon, \mu}$ the triple $(C, C_\varepsilon^0, C_\mu^1)$. This is an annulus with oriented pointed boundary. By definition,

$$\partial C_{\varepsilon, \mu} = (\varepsilon C_\varepsilon^0) \cup (\mu C_\mu^1).$$

The homotopy class of a map $g : C_{\varepsilon, \mu} \rightarrow X$ is determined by the homotopy classes $\alpha, \beta \in \pi$ represented by the loops $g|_{C_\varepsilon^0}$ and $g|_{s \times [0, 1]}$, respectively. Here the interval $[0, 1]$ is oriented from 0 to 1. Note that the loop $g|_{C_\mu^1}$ represents $(\beta^{-1} \alpha^{-\varepsilon} \beta)^\mu$. We denote by $C_{\varepsilon, \mu}(\alpha; \beta)$ the annulus $C_{\varepsilon, \mu}$ endowed with the map to X corresponding to the pair $\alpha, \beta \in \pi$.

Let D be an oriented 2-disc with two holes. Denote the boundary components of D by Y, Z, T and provide them with base points y, z, t , respectively. For any signs $\varepsilon, \mu, \nu = \pm$ we denote by $D_{\varepsilon, \mu, \nu}$ the tuple $(D, Y_\varepsilon, Z_\mu, T_\nu)$. This is a 2-disc with two holes with oriented pointed boundary. By definition,

$$\partial D_{\varepsilon, \mu, \nu} = (\varepsilon Y_\varepsilon) \cup (\mu Z_\mu) \cup (\nu T_\nu).$$

To analyse the homotopy classes of maps $D_{\varepsilon, \mu, \nu} \rightarrow X$, we fix two proper embedded arcs ty and tz in D leading from t to y, z and mutually disjoint except in the endpoint t . To every map $g : D_{\varepsilon, \mu, \nu} \rightarrow X$ we assign the homotopy classes of the loops $g|_{Y_\varepsilon}, g|_{Z_\mu}, g|_{ty}, g|_{tz}$. This establishes a bijective correspondence between the set of homotopy classes of maps $D_{\varepsilon, \mu, \nu} \rightarrow X$ and π^4 . For any $\alpha, \beta, \rho, \delta \in \pi$ denote by $D_{\varepsilon, \mu, \nu}(\alpha, \beta; \rho, \delta)$ the disc $D_{\varepsilon, \mu, \nu}$ endowed with the map to X corresponding to the tuple $\alpha, \beta, \rho, \delta$. Note that the loops $g|_{Y_\varepsilon}, g|_{Z_\mu}, g|_{T_\nu}$ represent the classes $\alpha, \beta, (\rho \alpha^{-\varepsilon} \rho^{-1} \delta \beta^{-\mu} \delta^{-1})^\nu$, respectively.

4.7. Algebra L . We provide the direct sum

$$L = \bigoplus_{\alpha \in \pi} L_\alpha$$

with the structure of an associative algebra as follows. The disc with two holes $D_{--+}(\alpha, \beta; 1, 1)$ is an X -cobordism between $(Y_-, \alpha) \cup (Z_-, \beta)$ and $(T_+, \alpha\beta)$. Note that the map $D \rightarrow X$ in question (considered up to homotopy) sends the intervals ty, tz into the base point $x \in X$. The corresponding homomorphism

$$\tau(D_{--+}(\alpha, \beta; 1, 1)) : L_\alpha \otimes L_\beta \rightarrow L_{\alpha\beta}$$

defines a K -bilinear multiplication in L by

$$(4.7.a) \quad ab = \tau(D_{--+}(\alpha, \beta; 1, 1))(a \otimes b) \in L_{\alpha\beta}$$

for $a \in L_\alpha, b \in L_\beta$. By a standard argument, this multiplication is associative. The key point is that under the gluing of $D_{--+}(\alpha, \beta; 1, 1)$ to $D_{--+}(\alpha\beta, \gamma; 1, 1)$ along an X -homeomorphism $(T_+, \alpha\beta) = (Y_-, \alpha\beta)$ we obtain the same X -cobordism as under the gluing of $D_{--+}(\beta, \gamma; 1, 1)$ to $D_{--+}(\alpha, \beta\gamma; 1, 1)$ along a homeomorphism $(T_+, \beta\gamma) = (Z_-, \beta\gamma)$.

The unit of L is constructed as follows. Let B_+ be an oriented 2-disc whose boundary is pointed and endowed with the orientation induced from B_+ . There is only one homotopy class of maps $B_+ \rightarrow X$. The corresponding homomorphism $\tau(B_+) : K \rightarrow L_1$ sends the unit $1 \in K$ into an element of L_1 , denoted 1_L . This element is a right unit in L because the gluing of B_+ to $D_{--+}(\alpha, 1; 1, 1)$ along an X -homeomorphism $\partial B_+ = Z_-$ yields the annulus $C_{-+}(\alpha; 1)$ and axioms (1.2.6, 1.2.7) apply. Similarly, 1_L is a left unit of L .

4.8. Action of π and the form η . The group π acts on L as follows. For $\alpha, \beta \in \pi$, the annulus $C_{-+}(\alpha; \beta^{-1})$ is an X -cobordism between (C_-^0, α) and $(C_+^1, \beta\alpha\beta^{-1})$. Set

$$\varphi_\beta = \tau(C_{-+}(\alpha; \beta^{-1})) : L_\alpha \rightarrow L_{\beta\alpha\beta^{-1}}.$$

Observe that the gluing of $C_{-+}(\alpha; \beta^{-1})$ to $C_{-+}(\beta\alpha\beta^{-1}; \gamma^{-1})$ with $\gamma \in \pi$ yields $C_{-+}(\alpha; (\gamma\beta)^{-1})$. Axiom (1.2.6) implies that $\varphi_{\gamma\beta} = \varphi_\gamma \varphi_\beta$. By axiom (1.2.7), $\varphi_1 = \text{id}$.

The annulus $C_{--}(\alpha; 1)$ is an X -cobordism between $(C_-^0, \alpha) \cup (C_-^1, \alpha^{-1})$ and \emptyset . Set

$$\eta|_{L_\alpha \otimes L_{\alpha^{-1}}} = \tau(C_{--}(\alpha; 1)) : L_\alpha \otimes L_{\alpha^{-1}} \rightarrow K.$$

The direct sum of these pairings over all $\alpha \in \pi$ yields the form $\eta : L \otimes L \rightarrow K$.

4.9. Lemma. *The algebra L with action φ and form η is a crossed π -algebra.*

Proof. The existence of a unit was verified above. Let us verify the other axioms.

Verification of (3.1.1). Note that by axiom (1.2.7), $\tau(C_{-+}(\alpha; 1)) = \text{id}_{L_\alpha}$. The annulus $C_{-+}(\alpha; 1)$ can be obtained by gluing $C_{--}(\alpha; 1)$ to $C_{++}(\alpha^{-1}; 1)$ along a homeomorphism $(C_-^1, \alpha^{-1}) = (C_+^0, \alpha^{-1})$. The multiplicativity of τ implies that

$$(\tau(C_{--}(\alpha; 1)) \otimes \text{id}_{L_\alpha}) \circ (\text{id}_{L_\alpha} \otimes \tau(C_{++}(\alpha^{-1}; 1))) = \tau(C_{-+}(\alpha; 1)) = \text{id}_{L_\alpha}.$$

If K is a field then presenting the homomorphisms $\tau(C_{--}(\alpha; 1)) = \eta|_{L_\alpha \otimes L_{\alpha^{-1}}}$ and $\tau(C_{++}(\alpha^{-1}; 1)) : K \rightarrow L_{\alpha^{-1}} \otimes L_\alpha$ by matrices (with respect to certain bases of $L_\alpha, L_{\alpha^{-1}}$) we easily obtain that $\dim L_\alpha \leq \dim L_{\alpha^{-1}}$ and the matrix of the pairing $\eta|_{L_\alpha \otimes L_{\alpha^{-1}}}$ admits a right inverse. By symmetry, $\dim L_\alpha = \dim L_{\alpha^{-1}}$ and the latter pairing is non-degenerate. In the case of an arbitrary K one should use the argument given in [Tu, Section III.2].

Verification of (3.1.2). Let us prove that $\eta(ab, c) = \eta(a, bc)$ for any $a \in L_\alpha, b \in L_\beta, c \in L_\gamma$ where $\alpha, \beta, \gamma \in \pi$. If $\alpha\beta\gamma \neq 1$, then $\eta(ab, c) = 0 = \eta(a, bc)$. Assume that $\alpha\beta\gamma = 1$. Gluing the annulus $C_{--}(\alpha\beta; 1)$ to $D_{--+}(\alpha, \beta; 1, 1)$ along an X -homeomorphism $(C_-^0, \alpha\beta) = (T_+, \alpha\beta)$ we obtain $D_{---}(\alpha, \beta; 1, 1)$. Therefore

$$\eta(ab, c) = \tau(D_{---}(\alpha, \beta; 1, 1))(a \otimes b \otimes c)$$

where

$$\tau(D_{---}(\alpha, \beta; 1, 1)) \in \text{Hom}_K(L_\alpha \otimes L_\beta \otimes L_\gamma, K).$$

Gluing $C_{--}(\alpha; 1)$ and $D_{--+}(\beta, \gamma; 1, 1)$ along an X -homeomorphism $(C_-^1, \alpha^{-1}) = (T_+, \beta\gamma)$ we obtain $D_{---}(\beta, \gamma; 1, 1)$. Hence

$$\eta(a, bc) = \tau(D_{---}(\beta, \gamma; 1, 1))(b \otimes c \otimes a)$$

where

$$\tau(D_{---}(\beta, \gamma; 1, 1)) \in \text{Hom}_K(L_\beta \otimes L_\gamma \otimes L_\alpha, K).$$

It remains to observe that there is an X -homeomorphism $D_{---}(\alpha, \beta; 1, 1) \rightarrow D_{---}(\beta, \gamma; 1, 1)$ mapping the boundary components Y, Z, T of the first disc with holes onto the boundary components T, Y, Z of the second disc with holes, respectively. By axiom (1.2.4), $\eta(ab, c) = \eta(a, bc)$.

Verification of (3.2.1). Let us prove that $\varphi_\gamma(ab) = \varphi_\gamma(a)\varphi_\gamma(b)$ for any $a \in L_\alpha, b \in L_\beta$ where $\alpha, \beta, \gamma \in \pi$. Gluing $C_{-+}(\alpha\beta; \gamma^{-1})$ to $D_{--+}(\alpha, \beta; 1, 1)$ along an X -homeomorphism $(C_-^0, \alpha\beta) = (T_+, \alpha\beta)$, we obtain $D_{--+}(\alpha, \beta; \gamma, \gamma)$. Hence

$$\varphi_\gamma(ab) = \tau(D_{--+}(\alpha, \beta; \gamma, \gamma))(a \otimes b)$$

where

$$\tau(D_{--+}(\alpha, \beta; \gamma, \gamma)) \in \text{Hom}_K(L_\alpha \otimes L_\beta, L_{\gamma\alpha\beta\gamma^{-1}}).$$

Similarly, gluing $C_{-+}(\alpha; \gamma^{-1}) \cup C_{-+}(\beta; \gamma^{-1})$ to $D_{--+}(\gamma\alpha\gamma^{-1}, \gamma\beta\gamma^{-1}; 1, 1)$ along X -homeomorphisms

$$(C_+^1, \gamma\alpha\gamma^{-1}) = (Y_-, \gamma\alpha\gamma^{-1}) \quad \text{and} \quad (C_+^1, \gamma\beta\gamma^{-1}) = (Z_-, \gamma\beta\gamma^{-1}),$$

respectively, we obtain $D_{--+}(\alpha, \beta; \gamma, \gamma)$. Therefore

$$\varphi_\gamma(a) \varphi_\gamma(b) = \tau(D_{--+}(\alpha, \beta; \gamma, \gamma))(a \otimes b) = \varphi_\gamma(ab).$$

The proof of the identity $\eta(\varphi_\gamma(a), \varphi_\gamma(b)) = \eta(a, b)$ is similar.

Verification of (3.2.2). The Dehn twist along the circle $S^1 \times (1/2) \subset C_{-+}(\alpha; 1)$ yields an X -homeomorphism $C_{-+}(\alpha; 1) \rightarrow C_{-+}(\alpha; \alpha)$. Axiom (1.2.4) implies that $\tau(C_{-+}(\alpha; \alpha)) = \tau(C_{-+}(\alpha; 1)) = \text{id}$. Therefore $\varphi_\alpha|_{L_\alpha} = \text{id}$.

Verification of (3.2.3). Note first that for any $\rho, \delta \in \pi$ the homomorphism

$$\tau(D_{--+}(\alpha, \beta; \rho, \delta)) : L_\alpha \otimes L_\beta \rightarrow L_{\rho\alpha\rho^{-1}\delta\beta\delta^{-1}}$$

can be computed in terms of φ and multiplication in L :

$$(4.9.a) \quad \tau(D_{--+}(\alpha, \beta; \rho, \delta))(a \otimes b) = \varphi_\rho(a) \varphi_\delta(b).$$

This follows from the multiplicativity of τ using the splitting of $D_{--+}(\alpha, \beta; \rho, \delta)$ as a union of $D_{--+}(\alpha, \beta; 1, 1)$ with $C_{-+}(\alpha; \rho^{-1})$ and $C_{-+}(\beta; \delta^{-1})$.

Consider a self-homeomorphism f of the disc with two holes D which is the identity on T and which permutes (Y, y) and (Z, z) . We choose f so that $f(tz) = ty$ and $f(ty)$ is an embedded arc leading from t to z and homotopic to the product of four arcs $ty, \partial Y, (ty)^{-1}, tz$. An easy computation shows that f is an X -homeomorphism $D_{--+}(\alpha, \beta; 1, 1) \rightarrow D_{--+}(\beta, \alpha; 1, \beta^{-1})$. Axiom (1.2.4) implies that the homomorphisms

$$\tau(D_{--+}(\alpha, \beta; 1, 1)) : L_\alpha \otimes L_\beta \rightarrow L_{\alpha\beta}$$

and

$$\tau(D_{--+}(\beta, \alpha; 1, \beta^{-1})) : L_\beta \otimes L_\alpha \rightarrow L_{\alpha\beta}$$

are obtained from each other by the permutation of the two tensor factors. Therefore for any $a \in L_\alpha, b \in L_\beta$

$$ab = \tau(D_{--+}(\alpha, \beta; 1, 1))(a \otimes b) = \tau(D_{--+}(\beta, \alpha; 1, \beta^{-1}))(b \otimes a) = b \varphi_{\beta^{-1}}(a).$$

This is equivalent to (3.2.2).

Verification of (3.2.4). Fix an orientation of S^1 and consider the 2-torus $S^1 \times S^1$ with product orientation. Let $B \subset S^1 \times S^1$ be a closed embedded 2-disc disjoint from the loops $S^1 \times s$ and $s \times S^1$ where $s \in S^1$. Consider the punctured torus $H = (S^1 \times S^1) \setminus \text{Int} B$ with orientation induced from $S^1 \times S^1$. Let us provide the boundary circle $\partial H = \partial B$ with orientation opposite to the one induced from H . We choose a base point on ∂H and an arc $r \subset H$ joining this point to $s \times s \in H$. We can assume that r meets the loops $S^1 \times s$ and $s \times S^1$ only in its endpoint $s \times s$.

Consider a map $g : H \rightarrow X = K(\pi, 1)$ such that $g(r) = x \in X$ and the restrictions of g to $S^1 \times s, s \times S^1$ represent $\alpha, \beta \in \pi$, respectively. (The orientations of $S^1 \times s, s \times S^1$ are induced by the one of S^1 .) Then the loop $g|_{\partial H}$ represents $\alpha\beta\alpha^{-1}\beta^{-1}$. Now, the pair (H, g) is an X -cobordism between $(\partial H_-, g|_{\partial H_-})$ and \emptyset .

This gives a homomorphism $\tau(H, g) : L_{\alpha\beta\alpha^{-1}\beta^{-1}} \rightarrow K$. We claim that both sides of formula (3.2.a) are equal to $\tau(H, g)(c)$. This would imply (3.2.4).

The pair (H, g) can be obtained from $D_{--+}(\alpha\beta\alpha^{-1}\beta^{-1}, \alpha; 1, \beta)$ by gluing the boundary components (Z_-, α) and (T_+, α) along an X -homeomorphism. (The circles Z_- and T_+ give the loop $S^1 \times s \subset T$.) A standard argument in the theory of TQFT's shows that the homomorphism $\tau(H, g) : L_{\alpha\beta\alpha^{-1}\beta^{-1}} \rightarrow K$ is the partial trace of the homomorphism

$$\tau(D_{--+}(\alpha\beta\alpha^{-1}\beta^{-1}, \alpha; 1, \beta)) : L_{\alpha\beta\alpha^{-1}\beta^{-1}} \otimes L_\alpha \rightarrow L_\alpha.$$

For $d \in L_\alpha$, we have

$$\tau(D_{--+}(\alpha\beta\alpha^{-1}\beta^{-1}, \alpha; 1, \beta))(c \otimes d) = c \varphi_\beta(d).$$

Therefore $\tau(H, g)(c) = \text{Tr}(c \varphi_\beta : L_\alpha \rightarrow L_\alpha)$. Similarly, the pair (H, g) is obtained from $D_{--+}(\alpha\beta\alpha^{-1}\beta^{-1}, \beta; \alpha^{-1}, \alpha^{-1})$ by gluing the boundary components (Z_-, β) and (T_+, β) along an X -homeomorphism. (The circles Z_- and T_+ give the loop $s \times S^1 \subset T$.) Thus, $\tau(H, g) : L_{\alpha\beta\alpha^{-1}\beta^{-1}} \rightarrow K$ is the partial trace of the homomorphism

$$\tau(D_{--+}(\alpha\beta\alpha^{-1}\beta^{-1}, \beta; \alpha^{-1}, \alpha^{-1})) : L_{\alpha\beta\alpha^{-1}\beta^{-1}} \otimes L_\beta \rightarrow L_\beta.$$

For $d \in L_\beta$, we have

$$\tau(D_{--+}(\alpha\beta\alpha^{-1}\beta^{-1}, \beta; \alpha^{-1}, \alpha^{-1}))(c \otimes d) = \varphi_{\alpha^{-1}}(c) \varphi_{\alpha^{-1}}(d) = \varphi_{\alpha^{-1}}(cd).$$

Therefore $\tau(H, g)(c) = \text{Tr}(\varphi_{\alpha^{-1}}c : L_\beta \rightarrow L_\beta)$.

4.10. Functoriality. The construction of the underlying crossed algebra is functorial with respect to morphisms of HQFT's as defined in Section 1.2. Such a morphism $\rho : (A, \tau) \rightarrow (A', \tau')$ yields for each $\alpha \in \pi$ a K -linear homomorphism $\rho_\alpha : L_\alpha \rightarrow L'_\alpha$ where L (resp. L') is the underlying crossed algebra of (A, τ) (resp. of (A', τ')). The commutativity of the natural square diagrams associated with the cobordisms $D_{--+}(\alpha, \beta; 1, 1)$, B_+ , $C_{--}(\alpha; 1)$, and $C_{-+}(\alpha; \beta^{-1})$ imply that $\oplus_\alpha \rho_\alpha : L \rightarrow L'$ is an algebra homomorphism preserving the unit and the inner product and commuting with the action of π . This establishes the functoriality of the underlying crossed algebra.

4.11. The Verlinde formula. The arguments given at the end of Section 4.9 allow us to compute explicitly the values of τ on closed oriented X -surfaces. Note first that for any $\alpha, \beta \in \pi$ there is a unique element $h_{\alpha, \beta} \in L_{\beta\alpha\beta^{-1}\alpha^{-1}}$ such that

$$\eta(c, h_{\alpha, \beta}) = \text{Tr}(c \varphi_\beta : L_\alpha \rightarrow L_\alpha) = \text{Tr}(\varphi_{\alpha^{-1}}c : L_\beta \rightarrow L_\beta)$$

for any $c \in L_{\alpha\beta\alpha^{-1}\beta^{-1}}$. The element $h_{\alpha, \beta}$ can be geometrically described as follows. Let (H', g) be the same punctured X -torus (H, g) as in Section 4.9 with

opposite orientation of ∂H (so that it is induced from H). The pair (H', g) is an X -cobordism between \emptyset and $(\partial H', g|_{\partial H'})$. Then $h_{\alpha, \beta} = \tau(H', g)(1_K)$ where $\tau(H', g) : K \rightarrow L_{\beta\alpha\beta^{-1}\alpha^{-1}}$ is the homomorphism associated with (H', g) .

To compute $\tau(W)$ for a closed oriented X -surface $(W, g : W \rightarrow X)$ of genus n , we choose a point $w \in W$ and a system of generators $a_1, a_2, \dots, a_{2n} \in \pi_1(W, w)$ subject to the only relation $\prod_{r=1}^n a_{2r-1} a_{2r} a_{2r-1}^{-1} a_{2r}^{-1} = 1$. We deform g so that $g(w) = x$. Then g induces a group homomorphism $g_{\#} : \pi_1(W, w) \rightarrow \pi = \pi_1(X, x)$. Using a decomposition of W into n copies of H' , a disc with n holes, and a disc B_- (cf. Section 5.1 below) one can easily compute that

$$(4.11.a) \quad \tau(W, g) = \eta\left(\prod_{r=1}^n h_{g_{\#}(a_{2r}), g_{\#}(a_{2r-1})}, 1_L\right) \in K.$$

Note that the relation $\prod_{r=1}^n a_{2r-1} a_{2r} a_{2r-1}^{-1} a_{2r}^{-1} = 1$ yields $\prod_{r=1}^n h_{g_{\#}(a_{2r}), g_{\#}(a_{2r-1})} \in L_1$.

If the crossed π -algebra L underlying (A, τ) is semisimple then we can split each $h_{\alpha, \beta} \in L_{\beta\alpha\beta^{-1}\alpha^{-1}}$ as follows:

$$h_{\alpha, \beta} = \sum_{i \in \text{bas}(L)} h_{\alpha, \beta, i}$$

where $h_{\alpha, \beta, i} = i h_{\alpha, \beta} \in i L_{\beta\alpha\beta^{-1}\alpha^{-1}}$. Now we can rewrite (4.11.a) in the form

$$(4.11.b) \quad \tau(W, g) = \sum_{i \in \text{bas}(L)} \eta\left(\prod_{r=1}^n h_{g_{\#}(a_{2r}), g_{\#}(a_{2r-1}), i}, i\right).$$

This formula generalizes the well known Verlinde formula corresponding to $\pi = 1$: in this case $h_{1,1,i} = k_i i$ with $k_i \in K$ and formula (4.11.b) has the standard form $\tau(W) = \sum_{i \in \text{bas}(L)} \eta(i, i)(k_i)^n$.

If K is a field of characteristic 0, then by (3.6.a) the i -th summand on the right-hand side of (4.11.b) is 0 unless $g_{\#}(a_{2r-1} a_{2r} a_{2r-1}^{-1} a_{2r}^{-1})(i) = i$ for all $r = 1, \dots, n$.

Formula (4.11.a) extends to X -surfaces with boundary. Consider for simplicity an X -cobordism W with bottom base $\cup_{p=1}^m (C_p^-, \alpha_p \in \pi)$ as in Section 4.5 and empty top base. The homomorphism $\tau(W) : \bigotimes_{p=1}^m L_{\alpha_p} \rightarrow K$ is computed as follows. Choose a point $w \in \text{Int}W$ and deform g relative to ∂W so that $g(w) = x$. Now, connect w by disjoint (except in w) embedded arcs, d_p , to the base points of the circles $\{C_p^-\}$. Denote by c_p the element of $\pi_1(W, w)$ represented by the loop obtained as the product of d_p , the loop parametrizing C_p^- and the path inverse to d_p . The group $\pi_1(W, w)$ can be presented by $2n + m$ generators $a_1, a_2, \dots, a_{2n}, c_1, \dots, c_m$ and one relation

$$\prod_{r=1}^n a_{2r-1} a_{2r} a_{2r-1}^{-1} a_{2r}^{-1} \prod_{p=1}^m c_p = 1.$$

The paths d_p are mapped by g into loops in (X, x) representing certain $\{\gamma_p \in \pi\}_p$. Then for any $\{u_p \in L_{\alpha_p}\}_p$ we have

$$\tau(W)(\otimes_{p=1}^m u_p) = \eta\left(\prod_{r=1}^n h_{g_{\#}(a_{2r}), g_{\#}(a_{2r-1})} \prod_{p=1}^m \varphi_{\gamma_p}(u_p), 1_L\right).$$

4.12. Remark. We may consider generalized $(1+1)$ -dimensional HQFT's such that the homomorphism τ is associated only with surfaces of genus 0. These HQFT's correspond to generalized crossed algebras defined as the crossed algebras above but without axiom (3.2.4).

5. Proof of Theorems 4.1 and 4.3

Throughout this section we shall use the term X -surface for any 2-dimensional X -cobordism. When the underlying surface is a disc, an annulus or a disc with holes we call the X -surface an X -disc, an X -annulus, or an X -disc with holes, respectively.

5.1. Proof of Theorem 4.1: the bijectivity for morphisms. We show here that for any two $(1+1)$ -dimensional HQFT's $(A, \tau), (A', \tau')$ with target $X = K(\pi, 1)$ with underlying crossed algebras L, L' , the homomorphism

$$(5.1.a) \quad \text{Hom}((A, \tau), (A', \tau')) \rightarrow \text{Hom}(L, L')$$

constructed in Section 4.10 is bijective. The injectivity of this homomorphism is obvious since all 1-dimensional X -manifolds are disjoint unions of loops and therefore any two morphisms $(A, \tau) \rightarrow (A', \tau')$ coinciding on loops coincide on all 1-dimensional X -manifolds.

To establish the surjectivity of (5.1.a) we first show how to reconstruct (A, τ) (at least up to isomorphism) from the underlying crossed π -algebra $L = (L, \eta, \varphi)$.

It follows from the topological classification of surfaces that every compact oriented surface can be split along a finite set of disjoint simple loops into a union of discs with ≤ 2 holes, i.e., discs, annuli and discs with two holes. This implies that every X -surface W can be obtained by gluing from a finite collection of X -surfaces whose underlying surfaces are discs with ≤ 2 holes. Axioms of an HQFT imply that the vector $\tau(W)$ is determined by the values of τ on the discs with ≤ 2 holes. It remains to show that these values are completely determined by (L, η, φ) .

We begin by computing τ for annuli. Each X -annulus is X -homeomorphic either to $C_{-+}(\alpha; \beta)$, or to $C_{--}(\alpha; \beta)$, or to $C_{++}(\alpha; \beta)$ with $\alpha, \beta \in \pi$. (Note that C_{+-} is homeomorphic to C_{-+} .) By definition,

$$(5.1.b) \quad \tau(C_{-+}(\alpha; \beta))(a) = \varphi_{\beta^{-1}}(a)$$

for any $a \in L_{\alpha}$.

The annulus $C_{--}(\alpha; \beta)$ can be obtained by the gluing of two annuli $C_{-+}(\alpha; \beta)$ and $C_{--}(\beta^{-1}\alpha\beta; 1)$ along an X -homeomorphism $(C_+^1, \beta^{-1}\alpha\beta) = (C_-^0, \beta^{-1}\alpha\beta)$. Axiom (1.2.6) and the definition of η imply that

$$\tau(C_{--}(\alpha; \beta)) \in \text{Hom}_K(L_\alpha \otimes L_{\beta^{-1}\alpha^{-1}\beta}, K)$$

is computed by

$$(5.1.c) \quad \tau(C_{--}(\alpha; \beta))(a \otimes b) = \eta(\varphi_{\beta^{-1}}(a), b)$$

where $a \in L_\alpha$, $b \in L_{\beta^{-1}\alpha^{-1}\beta}$.

To compute the vector

$$\tau(C_{++}(\alpha; \beta)) \in \text{Hom}_K(K, L_\alpha \otimes L_{\beta^{-1}\alpha^{-1}\beta}) = L_\alpha \otimes L_{\beta^{-1}\alpha^{-1}\beta}$$

we present this vector as a finite sum $\sum_i a_i \otimes b_i$ where $a_i \in L_\alpha$ and $b_i \in L_{\beta^{-1}\alpha^{-1}\beta}$. Observe that the gluing of $C_{--}(\alpha^{-1}; 1)$ to $C_{++}(\alpha; \beta)$ along an X -homeomorphism $(C_-^1, \alpha) = (C_+^0, \alpha)$ yields $C_{-+}(\alpha^{-1}; \beta)$. The axioms of an HQFT and the computations above yield

$$(5.1.d) \quad \sum_i \eta(a, a_i) b_i = \varphi_{\beta^{-1}}(a)$$

for any $a \in L_{\alpha^{-1}}$. Since the restriction of η to $L_{\alpha^{-1}} \times L_\alpha$ is non-degenerate, equality (5.1.d) determines uniquely the vector $\tau(C_{++}(\alpha; \beta)) = \sum_i a_i \otimes b_i$.

There are two X -discs: B_+ where the orientation of the boundary is induced by the one in the disc and B_- where the orientation of the boundary is opposite to the one induced from the disc. The vector $\tau(B_+) = 1_L \in L_1$ is determined uniquely as the unit element of L . The X -disc B_- may be obtained by the gluing of B_+ and $C_{--}(1; 1)$ along $\partial B_+ = C^0$. Therefore $\tau(B_-) \in \text{Hom}_K(L_1, K)$ is determined uniquely by L . One can compute that $\tau(B_-)(a) = \eta(a, 1_L) = \eta(1_L, a)$ for any $a \in L_1$.

Each disc with two holes can be split along 3 disjoint simple loops parallel to its boundary components into a union of 3 annuli and a smaller disc with two holes. Choosing appropriate orientations of these 3 loops we obtain that any X -disc with two holes D splits as a union of 3 annuli and an X -disc with two holes X -homeomorphic to $D_{--+}(\alpha, \beta; 1, 1)$ for certain $\alpha, \beta \in \pi$. The homomorphism $\tau(D_{--+}(\alpha, \beta; 1, 1))$ is multiplication in L . The values of τ on the annuli are also determined by L by the arguments above. Now, the axioms of an HQFT allow us to recover $\tau(D)$ uniquely.

Now we can prove the surjectivity of (5.1.a). Every morphism of crossed π -algebras $\rho : L \rightarrow L'$ defines a linear homomorphism $A_M \rightarrow A_{M'}$ for any connected 1-dimensional X -manifold M . These homomorphisms extend to non-connected X -manifolds M by multiplicativity. We should show that the resulting family of K -linear homomorphisms $\{\rho_M : A_M \rightarrow A_{M'}\}_M$ makes the natural square diagrams associated with X -homeomorphisms and X -surfaces commutative. The

part concerning the homeomorphisms is obvious. As it was explained above, every X -surface can be obtained by gluing from a finite collection of X -surfaces of type B_+ , $C_{-+}(\alpha; \beta)$, $C_{--}(\alpha; 1)$, $C_{++}(\alpha; \beta)$, and $D_{--+}(\alpha, \beta; 1, 1)$. Therefore it suffices to check the commutativity of the square diagrams associated with these X -surfaces. For B_+ and $D_{--+}(\alpha, \beta; 1, 1)$ this follows from the assumption that $\rho : L \rightarrow L'$ is an algebra homomorphism preserving the unit. For the annuli $C_{-+}(\alpha; \beta)$, $C_{--}(\alpha; 1)$, $C_{++}(\alpha; \beta)$, this follows from the formulas (5.1.b) - (5.1.d) and the assumption that $\rho : L \rightarrow L'$ preserves the inner product and commutes with the action of π .

5.2. Proof of Theorem 4.1: the surjectivity for objects. Let $(L = \bigoplus_{\alpha \in \pi} L_\alpha, \eta, \varphi)$ be a crossed π -algebra. We shall realize it as the underlying algebra of a $(1+1)$ -dimensional HQFT (A, τ) with target $X = K(\pi, 1)$.

We associate with every 1-dimensional X -manifold a K -module as in Section 4.5. It remains to define the homomorphisms τ for 2-dimensional X -cobordisms. The construction of τ goes in nine steps.

Step 1. We define the homomorphism τ for X -annuli using formulas (5.1.b) - (5.1.d). To establish the topological invariance, note that the X -homeomorphisms of X -annuli are generated by (i) the Dehn twists $C_{\varepsilon, \mu}(\alpha; \beta) \rightarrow C_{\varepsilon, \mu}(\alpha; \beta\alpha)$ along the circle $S^1 \times (1/2)$ where $\varepsilon, \mu = \pm$ and (ii) the homeomorphisms $C_{\varepsilon, \varepsilon}(\alpha; \beta) \rightarrow C_{\varepsilon, \varepsilon}(\beta^{-1}\alpha^{-1}\beta; \beta^{-1})$ permuting the boundary components of the annulus and preserving the arc $s \times [0, 1]$ (with $s \in S^1$) as a set.

The invariance of τ under the Dehn twists follows from the equalities $\varphi_{\beta\alpha}|_{L_\alpha} = \varphi_\beta\varphi_\alpha|_{L_\alpha} = \varphi_\beta|_{L_\alpha}$.

The homomorphism τ defined by (5.1.c) is invariant under the homeomorphism (ii) with $\varepsilon = -$ because

$$\begin{aligned} \tau(C_{--}(\alpha; \beta))(a \otimes b) &= \eta(\varphi_{\beta^{-1}}(a), b) = \eta(a, \varphi_\beta(b)) = \eta(\varphi_\beta(b), a) \\ &= \tau(C_{--}(\beta^{-1}\alpha^{-1}\beta; \beta^{-1}))(b \otimes a). \end{aligned}$$

To prove that the homomorphism $\tau(C_{++})$ defined by (5.1.d) is invariant under the homeomorphism $C_{++}(\alpha; \beta) \rightarrow C_{++}(\beta^{-1}\alpha^{-1}\beta; \beta^{-1})$ described above, it suffices to deduce from (5.1.d) that for any $b \in L_{\beta^{-1}\alpha\beta}$,

$$(5.2.a) \quad \sum_i \eta(b, b_i) a_i = \varphi_\beta(b).$$

For any $a \in L_{\alpha^{-1}}$, we have

$$\begin{aligned} \eta(a, \varphi_\beta(b)) &= \eta(\varphi_{\beta^{-1}}(a), b) = \eta(b, \varphi_{\beta^{-1}}(a)) \\ &= \eta(b, \sum_i \eta(a, a_i) b_i) = \sum_i \eta(a, a_i) \eta(b, b_i) = \eta(a, \sum_i \eta(b, b_i) a_i). \end{aligned}$$

Now, the non-degeneracy of η implies (5.2.a).

Step 2. We check now axiom (1.2.6) in the case where W, W_0 and W_1 are annuli. It suffices to consider the case where the annulus $C_{\varepsilon, \mu}(\alpha; \beta)$ is glued to $C_{-\mu, \nu}(\gamma; \delta)$ along an X -homeomorphism $(C_{\mu}^1, (\beta^{-1}\alpha^{-\varepsilon}\beta)^{\mu}) = (C_{-\mu}^0, \gamma)$. Note that the gluing is possible only if $\gamma = (\beta^{-1}\alpha^{-\varepsilon}\beta)^{\mu}$; the result of the gluing is the annulus $C_{\varepsilon, \nu}(\alpha; \beta\delta)$. We consider 8 cases depending on the values of $\varepsilon, \mu, \nu = \pm$ and indicate the key argument implying (1.2.6); the details are left to the reader.

Cases (1) and (2): $\varepsilon = -, \mu = +, \nu = \pm$. Use that $\varphi_{\delta^{-1}}\varphi_{\beta^{-1}} = \varphi_{(\beta\delta)^{-1}}$.

Case (3): $\varepsilon = \mu = \nu = -$. This case follows from Case (2) by permuting the annuli under gluing: the gluing of C_{--} to C_{+-} along an X -homeomorphism $C_{-}^1 = C_{+}^0$ is the same operation as the gluing of C_{-+} to C_{--} along an X -homeomorphism $C_{+}^1 = C_{-}^0$.

Case (4): $\varepsilon = \mu = -, \nu = +$. We should prove that

$$(\tau(C_{--}(\alpha; \beta)) \otimes \text{id}_{L_{\delta^{-1}\gamma^{-1}\delta}})(\text{id}_{L_{\alpha}} \otimes \tau(C_{++}(\gamma; \delta))) = \tau(C_{-+}(\alpha; \beta\delta)).$$

By definition,

$$\tau(C_{++}(\gamma; \delta)) = \sum_i c_i \otimes d_i \in L_{\gamma} \otimes L_{\delta^{-1}\gamma^{-1}\delta}$$

where

$$(5.2.b) \quad \sum_i \eta(e, c_i) d_i = \varphi_{\delta^{-1}}(e)$$

for any $e \in L_{\gamma^{-1}}$. Note that $\gamma = \beta^{-1}\alpha^{-1}\beta$. For $a \in L_{\alpha}$, we have

$$\begin{aligned} & (\tau(C_{--}(\alpha; \beta)) \otimes \text{id}_{L_{\delta^{-1}\gamma^{-1}\delta}})(\text{id}_{L_{\alpha}} \otimes \tau(C_{++}(\gamma; \delta)))(a) \\ &= \sum_i (\tau(C_{--}(\alpha; \beta)) \otimes \text{id}_{L_{\delta^{-1}\gamma^{-1}\delta}})(a \otimes c_i \otimes d_i) \\ &= \sum_i \eta(\varphi_{\beta^{-1}}(a), c_i) d_i = \varphi_{\delta^{-1}}\varphi_{\beta^{-1}}(a) = \varphi_{(\beta\delta)^{-1}}(a) = \tau(C_{-+}(\alpha; \beta\delta))(a). \end{aligned}$$

Case (5): $\varepsilon = \mu = \nu = +$. Use that formula (5.1.d) implies the equality $\sum_i \eta(a, a_i) \varphi_{\delta^{-1}}(b_i) = \varphi_{(\beta\delta)^{-1}}(a)$.

Cases (6) and (7): $\varepsilon = +, \mu = \pm, \nu = -$. These cases follow from Cases (4) and (1) by permuting the annuli under gluing.

Case (8): $\varepsilon = +, \mu = -, \nu = +$. Use the same expression for $\tau(C_{++}(\gamma; \delta))$ as in Case (4) and the equalities

$$\sum_i \eta(e, \varphi_{\beta}(c_i)) d_i = \sum_i \eta(\varphi_{\beta^{-1}}(e), c_i) d_i = \varphi_{(\beta\delta)^{-1}}(e)$$

where $e \in L_{\beta\gamma^{-1}\beta^{-1}}$.

Step 3. At steps 3 - 5 we define τ for discs with two holes $D_{\varepsilon, \mu, \nu}(\alpha, \beta; \rho, \delta)$ where $\alpha, \beta, \rho, \delta \in \pi$.

For an X -disc with two holes $D = D_{--+}(\alpha, \beta; \rho, \delta)$, cf. Section 4.6, we define $\tau(D) \in \text{Hom}_K(L_\alpha \otimes L_\beta, L_{\rho\alpha\rho^{-1}\delta\beta\delta^{-1}})$ by $\tau(D)(a \otimes b) = \varphi_\rho(a) \varphi_\delta(b)$ where $a \in L_\alpha, b \in L_\beta$. The topological invariance of $\tau(D)$ follows from axioms (3.2.2, 3.2.3) and the fact that any self-homeomorphism of $D = D_{--+}$ is isotopic to a composition of Dehn twists in annuli neighborhoods of the circles $Y, Z \subset \partial D$ and the homeomorphisms $f^{\pm 1} : D \rightarrow D$ introduced in the proof of Lemma 4.9.

Axiom (1.2.6) holds for any gluing of an annulus of type C_{-+} to D_{--+} : if the gluing is performed along an X -homeomorphism $C_+^1 = Y_-$ or $C_+^1 = Z_-$ then this follows from the identity $\varphi_\alpha \varphi_\beta = \varphi_{\alpha\beta}$; if the gluing is performed along $C_-^0 = T_+$ then this follows from the assumption that each φ_α is an algebra homomorphism.

Step 4. Consider an X -disc with two holes $D = D_{---}(\alpha, \beta; \rho, \delta)$ and set

$$\gamma = \delta\beta^{-1}\delta^{-1}\rho\alpha^{-1}\rho^{-1}.$$

We define $\tau(D) \in \text{Hom}_K(L_\alpha \otimes L_\beta \otimes L_\gamma, K)$ by $\tau(D)(a \otimes b \otimes c) = \eta(\varphi_\rho(a) \varphi_\delta(b), c)$ where $a \in L_\alpha, b \in L_\beta, c \in L_\gamma$. This definition results immediately from axiom (1.2.6) if we present D as the result of a gluing of $D_{--+}(\alpha, \beta; \rho, \delta)$ to $C_{--}(\gamma^{-1}; 1)$ along $(T_+, \gamma^{-1}) = (C_-^0, \gamma^{-1})$.

Let us verify the topological invariance of $\tau(D)$. Consider an X -homeomorphism $h : D_{---} \rightarrow D_{---}$ which maps $(Y, y), (Z, z), (T, t)$ onto $(Z, z), (T, t), (Y, y)$, respectively. We choose h so that the arc tz is mapped onto $yt = (ty)^{-1}$ and the arc ty is mapped onto an embedded arc leading from y to z and homotopic to the product of the arcs yt and tz . An easy computation shows that h is an X -homeomorphism $D_{---}(\alpha, \beta; \rho, \delta) \rightarrow D_{---}(\gamma, \alpha; \delta^{-1}, \delta^{-1}\rho)$. The invariance of τ under h follows from the equalities

$$\eta(\varphi_{\delta^{-1}}(c) \varphi_{\delta^{-1}\rho}(a), b) = \eta(c \varphi_\rho(a), \varphi_\delta(b)) = \eta(c, \varphi_\rho(a) \varphi_\delta(b)) = \eta(\varphi_\rho(a) \varphi_\delta(b), c).$$

It is clear that any self- X -homeomorphism of D may be presented as a composition of $h^{\pm 1}$ with a self-homeomorphism of D preserving all boundary components set-wise. Therefore it remains only to check the topological invariance of $\tau(D)$ under self-homeomorphisms of D preserving the boundary components. Such homeomorphisms (considered up to isotopy) are compositions of Dehn twists in annuli neighborhoods of Y, Z, T . Invariance of $\tau(D)$ under such Dehn twists follows from the already established topological invariance of the homomorphisms $\tau(D_{--+}(\alpha, \beta; \rho, \delta))$ and $\tau(C_{--}(\gamma^{-1}; 1))$.

Axiom (1.2.6) holds for any gluing of an annulus of type C_{-+} to D_{---} (such a gluing produces again D_{---}). If the gluing is performed along Y or Z then this follows from the identity $\varphi_\alpha \varphi_\beta = \varphi_{\alpha\beta}$. The existence of a self-homeomorphism of D_{---} mapping T onto Y shows that the claim holds also for the gluings along T .

Step 5. Now we define τ for $D = D_{++-}(\alpha, \beta; \rho, \delta)$. We can obtain D by gluing three annuli $C_{++}(\alpha; 1), C_{++}(\beta; 1), C_{--}(\gamma; 1)$ with $\gamma = \rho\alpha^{-1}\rho^{-1}\delta\beta^{-1}\delta^{-1}$ to $D_{--+}(\alpha^{-1}, \beta^{-1}; \rho, \delta)$ along X -homeomorphisms

$$(C_+^1, \alpha^{-1}) = (Y_-, \alpha^{-1}), (C_+^1, \beta^{-1}) = (Z_-, \beta^{-1}), (C_-^0, \gamma) = (T_+, \gamma),$$

respectively. These three annuli form a regular neighborhood of ∂D in D . Axiom (1.2.6) determines $\tau(D)$. Its topological invariance follows from the topological invariance of the values of τ for $D_{--+}(\alpha^{-1}, \beta^{-1}; \rho, \delta)$ and for the three annuli in question and the following obvious fact: any self-homeomorphism of D is isotopic to a homeomorphism mapping a given regular neighborhood of ∂D onto itself.

Similarly, we can obtain $D_{+++}(\alpha, \beta; \rho, \delta)$ by gluing 3 annuli of type $C_{++}(\dots; 1)$ to $D_{---}(\alpha^{-1}, \beta^{-1}; \rho, \delta)$. Axiom (1.2.6) determines $\tau(D_{+++}(\alpha, \beta; \rho, \delta))$ in a topologically invariant way.

Step 6. We check now axiom (1.2.6) for a gluing of an X -annulus $C_{\varepsilon^0, \varepsilon^1}$ to an X -disc with two holes $D_{\varepsilon, \mu, \nu}$. By the topological invariance of τ , it is enough to consider the gluings performed along an X -homeomorphism $C_{\varepsilon^0}^0 = T_\nu$ so that $\varepsilon^0 = -\nu$. We have 16 cases corresponding to different signs $\varepsilon^1, \varepsilon, \mu, \nu$. The cases where $\varepsilon^0 = -\varepsilon^1$ and the triple ε, μ, ν contains at least two minuses were considered at Steps 3 and 4. The cases where $\varepsilon = \mu$ are checked one by one using directly the definitions and the properties of τ established above, specifically, axiom (1.2.6) for annuli (Step 2). The key argument in all these cases is that the tensor contractions along different tensor factors commute. The case $\varepsilon = -, \mu = +$ reduces to $\varepsilon = +, \mu = -$ by the topological invariance. Assume that $\varepsilon = +, \mu = -$. If $\nu = +, \varepsilon^1 = +$ then (1.2.6) follows again from definitions. The remaining three cases ($\nu = +, \varepsilon^1 = -$), ($\nu = -, \varepsilon^1 = \pm$) can be deduced from the following multiplicativity of τ .

Let $\alpha, \beta, \rho, \delta, \Delta, \sigma \in \pi$. Set $\gamma = \rho\alpha\rho^{-1}\delta\beta\delta^{-1}$. Observe that the gluing of $D_{--+}(\alpha, \beta; \rho, \delta)$ to $C_{--}(\gamma; \Delta)$ along $(T_+, \gamma) = (C_-^0, \gamma)$ yields the X -disc with two holes $D = D_{---}(\alpha, \beta; \Delta^{-1}\rho, \Delta^{-1}\delta)$. The same X -disc with two holes D is obtained by gluing $\tilde{D} = D_{-+-}(\alpha, \sigma^{-1}\beta^{-1}\sigma; \Delta^{-1}\rho, \Delta^{-1}\delta\sigma)$ to $C_{--}(\beta; \sigma)$ along $(Z_+, \sigma^{-1}\beta^{-1}\sigma) = (C_-^1, \sigma^{-1}\beta^{-1}\sigma)$. This allows us to compute

$$\tau(D) \in \text{Hom}_K(L_\alpha \otimes L_\beta \otimes L_{\Delta^{-1}\gamma^{-1}\Delta}, K)$$

applying (1.2.6) to these two splittings of D . We claim that these two computations give the same result. The first splitting implies that for any $a \in L_\alpha, b \in L_\beta, c \in L_{\Delta^{-1}\gamma^{-1}\Delta}$,

$$\begin{aligned} \tau(D)(a \otimes b \otimes c) &= \tau(C_{--}(\gamma; \Delta))(\tau(D_{--+}(\alpha, \beta; \rho, \delta))(a \otimes b) \otimes c) \\ &= \eta(\varphi_{\Delta^{-1}}(\varphi_\rho(a)\varphi_\delta(b)), c). \end{aligned}$$

To use the second splitting we first observe that \tilde{D} is X -homeomorphic to

$$D_{--+}(\Delta^{-1}\gamma^{-1}\Delta, \alpha; \sigma^{-1}\delta^{-1}\Delta, \sigma^{-1}\delta^{-1}\rho)$$

via a homeomorphism mapping the boundary components Y, Z, T onto Z, T, Y , respectively. Therefore applying (1.2.6) to the second splitting of D we obtain

$$\tau(D)(a \otimes b \otimes c) = \tau(C_{--}(\beta; \sigma))(b \otimes \tau(\tilde{D})(c \otimes a))$$

$$\begin{aligned}
&= \eta(\varphi_{\sigma^{-1}}(b), \varphi_{\sigma^{-1}\delta^{-1}\Delta}(c) \varphi_{\sigma^{-1}\delta^{-1}\rho}(a)) = \eta(b, \varphi_{\delta^{-1}\Delta}(c) \varphi_{\delta^{-1}\rho}(a)) \\
&= \eta(\varphi_{\delta^{-1}\Delta}(c) \varphi_{\delta^{-1}\rho}(a), b) = \eta(\varphi_{\delta^{-1}\Delta}(c), \varphi_{\delta^{-1}\rho}(a)b) = \eta(\varphi_{\delta^{-1}\rho}(a)b, \varphi_{\delta^{-1}\Delta}(c)) \\
&= \eta(\varphi_{\Delta^{-1}\rho}(a) \varphi_{\Delta^{-1}\delta}(b), c) = \eta(\varphi_{\Delta^{-1}}(\varphi_{\rho}(a) \varphi_{\delta}(b)), c).
\end{aligned}$$

This completes the check of consistency and implies (1.2.6) for any gluing of an X -annulus to an X -disc with two holes.

Step 7. Now we define τ for the X -discs B_+ and B_- (see Section 5.1 for notation). Set $\tau(B_+) = 1_L \in L_1$. Axiom (1.2.6) for a gluing of B_+ to an X -annulus of type C_{-+} follows from the equality $\varphi_{\beta}(1_L) = 1_L$ for $\beta \in \pi$. Axiom (1.2.6) for a gluing of B_+ to an X -disc with two holes of type D_{--+} follows from the equalities $1_L a = a 1_L = a$ for any $a \in L$. This and the definition of τ for X -discs with two holes of types D_{---} or D_{-++} imply (1.2.6) for any gluing of B_+ to such discs with holes.

The disc B_- can be obtained by the gluing of B_+ and $C_{--}(1;1)$ along $\partial B_+ = C_-^0$. This determines $\tau(B_-) \in \text{Hom}_K(L_1, K)$. Axiom (1.2.6) for a gluing of B_- to X -discs with ≤ 2 holes follows from the already established properties of the gluings of $C_{--}(1;1)$ and B_+ .

Step 8. Now we define $\tau(W)$ for any connected X -surface $(W, g : W \rightarrow X)$. By a splitting system of loops on W we mean a finite set of disjoint embedded circles $\alpha_1, \dots, \alpha_N \subset W$ which split W into a union of discs with ≤ 2 holes. We provide each α_i with an orientation and a base point x_i . Replacing if necessary g by a homotopic map we can assume that $g(x_i) = x \in X$ for all i . The discs with holes obtained by the splitting of W along $\cup_i \alpha_i$ endowed with the restriction of g are X -surfaces. Axiom (1.2.6) determines $\tau(W)$ from the values of τ on these discs with holes.

We claim that $\tau(W)$ does not depend on the choice of orientations and base points on $\alpha_1, \dots, \alpha_N$. The proof is as follows. Choose $i \in \{1, \dots, N\}$ and set $\alpha = \alpha_i$. Let D_1, D_2 be the discs with holes attached to α from two sides (they may coincide). Let C be a regular neighborhood of α in D_1 . Clearly, C is an annulus in D_1 bounded by α and a parallel loop $\tilde{\alpha}$. Consider the discs with holes $\tilde{D}_1 = \overline{D_1 \setminus C}$ and $\tilde{D}_2 = \overline{D_2 \cup C}$. We provide $\tilde{\alpha}$ with a base point \tilde{x} and the orientation opposite to the one of α . We deform g in a small neighborhood of \tilde{x} so that $g(\tilde{x}) = x$. In these way the surfaces $C, \tilde{D}_1, \tilde{D}_2$ acquire the structure of X -surfaces. It follows from the properties of τ established above that

$$*_\alpha(\tau(D_1) \otimes \tau(D_2)) = *_\alpha *_\alpha(\tau(\tilde{D}_1) \otimes \tau(C) \otimes \tau(D_2)) = *_\alpha(\tau(\tilde{D}_1) \otimes \tau(\tilde{D}_2))$$

where $*_\alpha$ denotes the tensor contraction corresponding to the gluing along α . Thus, replacing α with $\tilde{\alpha}$ in our splitting system of loops we do not change $\tau(W)$. This implies that $\tau(W)$ does not depend on the choice of orientations and base points on $\alpha_1, \dots, \alpha_N$. A similar argument shows that $\tau(W)$ does not depend on the choice of g in its homotopy class (relative to the base points on ∂W).

We claim that the homomorphism $\tau(W)$ does not depend on the choice of a splitting system of loops on W . The crucial argument is provided by the fact (see

[HT]) that any two splitting systems of loops on W are related by the following transformations:

- (i) isotopy in W ;
- (ii) adding to a splitting system of loops $\{\alpha_1, \dots, \alpha_N\}$ a simple loop $\alpha \subset W \setminus \cup_i \alpha_i$;
- (iii) deleting a loop from a splitting system of loops, provided the remaining loops form a splitting system;
- (iv) replacing one of the loops α_i of a splitting system adjacent to two different discs with two holes D_1, D_2 by a simple loop lying in $\text{Int}D_1 \cup \text{Int}D_2 \cup \alpha_i$, meeting α_i transversally in two points and splitting both D_1 and D_2 into annuli;
- (v) replacing one of the loops of a splitting system by a simple loop meeting it transversally in one point and disjoint from the other loops.

We should check the invariance of $\tau(W)$ under these transformations. The invariance of $\tau(W)$ under isotopy is obvious. Consider the transformation (ii). The loop α lies in a disc with ≤ 2 holes obtained by the splitting of W along $\cup_i \alpha_i$. The loop α splits this disc with ≤ 2 holes into a union of a smaller disc with ≤ 2 holes and an annulus or a disc (without holes). Therefore the invariance of $\tau(W)$ under (ii) follows from the already established multiplicativity of τ under the gluings of a disc with ≤ 2 holes to an annulus or a disc. The transformation (iii) is inverse to (ii) and the same argument applies.

The associativity of multiplication in L yields two equivalent expressions for the value of τ for a disc with 3 holes; these expressions are obtained from two splittings of the disc with 3 holes as a union of two discs with 2 holes (cf. Section 4.7). Since we are free to choose orientations of the loops in a splitting system, we can reduce the invariance of $\tau(W)$ under the transformation (iv) to this model case. Note that in order to have the maps to X as in the model case we can use transformations (ii) to add additional annuli to the splitting. Similarly, the invariance of $\tau(W)$ under the transformation (v) follows from axiom (3.2.4) (cf. the end of the proof of Lemma 4.9).

The topological invariance of $\tau(W)$ follows from the topological invariance of τ for discs with ≤ 2 holes and the fact that any homeomorphism of connected surfaces maps a splitting system of loops onto a splitting system of loops.

Step 9. We have defined τ for connected X -surfaces. We extend τ to arbitrary X -surfaces by (1.2.5). It follows directly from definitions that (A, τ) is an HQFT. (To prove (1.2.6) we compute $\tau(W)$ using a splitting system of loops containing $N = N' \subset W$.)

5.3. Last claim of Theorem 4.1. The last claim of Theorem 4.1 is obvious for direct sums, tensor products, and pull-backs. For the duality and transfer, this claim is a nice computational exercise.

5.4. Proof of Theorem 4.3. Theorem 4.3 results from Theorems 3.6, 3.7, 4.1 and the fact that the underlying crossed algebra of a $(1+1)$ -dimensional primitive cohomological HQFT is the crossed algebra defined in Section 3.3.

6. Hermitian and unitary crossed algebras

In this section we assume that K has a ring involution $k \mapsto \bar{k} : K \rightarrow K$.

6.1. Hermitian and unitary crossed π -algebras. Let π be a group. A *Hermitian crossed π -algebra* is a crossed π -algebra $(L = \bigoplus_{\alpha \in \pi} L_\alpha, \eta, \varphi)$ over K endowed with an involutive antilinear antiautomorphism $a \mapsto \bar{a} : L \rightarrow L$ which commutes with the action of π , transforms η into $\bar{\eta}$ and sends each L_α into $L_{\alpha^{-1}}$. This definition implies the identities

$$(6.1.a) \quad \bar{\bar{a}} = a, \quad \overline{ka} = \bar{k}\bar{a}, \quad \overline{ab} = \bar{b}\bar{a}, \quad \varphi_\beta(\bar{a}) = \overline{\varphi_\beta(a)}, \quad \eta(\bar{a}, \bar{b}) = \overline{\eta(a, b)}$$

for any $a, b \in L, k \in K, \beta \in \pi$. It follows from these conditions that $\overline{1_L} = 1_L$.

Observe that for any $a \in L$,

$$\overline{\eta(a, \bar{a})} = \eta(\bar{a}, a) = \eta(a, \bar{a}).$$

In particular, if $K = \mathbf{C}$ with usual complex conjugation then $\eta(a, \bar{a}) \in \mathbf{R}$ for all a . If additionally, $\eta(a, \bar{a}) > 0$ for all non-zero $a \in L$, then we say that L is *unitary*.

It is easy to check that direct sums, tensor products, pull-backs, duals, and transfers of Hermitian (resp. unitary) crossed algebras are again Hermitian (resp. unitary) crossed algebras. We can define a category, $HQ_2(\pi)$ (resp. $UQ_2(\pi)$), whose objects are Hermitian (resp. unitary) crossed π -algebras. The morphisms are defined as in Section 3.2 with the additional condition that they commute with the involutions.

6.2. Examples. Let $\{\theta_{\alpha, \beta} \in S\}_{\alpha, \beta \in \pi}$ be a normalized 2-cocycle of the group π with values in the multiplicative group $S = \{k \in K \mid k\bar{k} = 1_K\}$. We introduce a Hermitian structure on the crossed π -algebra $L = \bigoplus_{\alpha \in \pi} Kl_\alpha$ constructed in Section 3.3. For all $\alpha \in \pi$, set

$$\bar{l}_\alpha = \overline{\theta_{\alpha, \alpha^{-1}}} l_{\alpha^{-1}} = (\theta_{\alpha, \alpha^{-1}})^{-1} l_{\alpha^{-1}}.$$

This extends uniquely to an antilinear homomorphism $a \mapsto \bar{a} : L \rightarrow L$. We claim that it satisfies (6.1.a). The involutivity follows from the equalities

$$\theta_{\alpha, \alpha^{-1}} \overline{\theta_{\alpha^{-1}, \alpha}} = \theta_{\alpha, \alpha^{-1}} \overline{\theta_{\alpha, \alpha^{-1}}} = 1_K$$

where we use the identity $\theta_{\alpha^{-1}, \alpha} = \theta_{\alpha, \alpha^{-1}}$. Similarly,

$$\begin{aligned} \eta(\bar{l}_\alpha, \bar{l}_{\alpha^{-1}}) &= \eta(\overline{\theta_{\alpha, \alpha^{-1}}} l_{\alpha^{-1}}, \overline{\theta_{\alpha^{-1}, \alpha}} l_\alpha) = \overline{\theta_{\alpha, \alpha^{-1}}} \overline{\theta_{\alpha^{-1}, \alpha}} \eta(l_{\alpha^{-1}}, l_\alpha) \\ &= \overline{\theta_{\alpha, \alpha^{-1}}} \overline{\theta_{\alpha^{-1}, \alpha}} \theta_{\alpha^{-1}, \alpha} = \overline{\theta_{\alpha, \alpha^{-1}}} = \overline{\eta(l_\alpha, l_{\alpha^{-1}})}. \end{aligned}$$

To prove that this involution is an antiautomorphism we should check that $\overline{\bar{l}_\alpha l_\beta} = \bar{l}_\beta \bar{l}_\alpha$ for any $\alpha, \beta \in \pi$. This is equivalent to

$$(6.2.a) \quad (\theta_{\alpha, \beta} \theta_{\alpha\beta, (\alpha\beta)^{-1}})^{-1} = (\theta_{\alpha, \alpha^{-1}} \theta_{\beta, \beta^{-1}})^{-1} \theta_{\beta^{-1}, \alpha^{-1}}.$$

To prove this formula, set $\gamma = \beta^{-1}$ in (3.3.a). This yields $\theta_{\alpha\beta, \beta^{-1}} = \theta_{\beta, \beta^{-1}}(\theta_{\alpha, \beta})^{-1}$. (We use that $\theta_{\alpha, 1} = 1$, see Section 3.3). Now, we replace α, β, γ in (3.3.a) with $\alpha\beta, \beta^{-1}, \alpha^{-1}$, respectively. Substituting in the resulting formula the expression $\theta_{\alpha\beta, \beta^{-1}} = \theta_{\beta, \beta^{-1}}(\theta_{\alpha, \beta})^{-1}$, we obtain a formula equivalent to (6.2.a). It remains to check the identity $\varphi_\beta(\overline{l_\alpha}) = \overline{\varphi_\beta(l_\alpha)}$. Note first that $l_\alpha \overline{l_\alpha} = 1$. Therefore $\varphi_\beta(l_\alpha) \varphi_\beta(\overline{l_\alpha}) = 1$. This characterizes $\overline{\varphi_\beta(l_\alpha)}$ as the (unique) element of $L_{\beta\alpha^{-1}\beta^{-1}}$ inverse to $\varphi_\beta(l_\alpha)$. We claim that $\overline{\varphi_\beta(l_\alpha)} \in L_{\beta\alpha^{-1}\beta^{-1}}$ is also inverse to $\varphi_\beta(l_\alpha)$. By definition, $\varphi_\beta(l_\alpha) = y l_{\beta\alpha\beta^{-1}}$ where $y \in K$ is determined from the equation $\varphi_\beta(l_\alpha) l_\beta = l_\beta l_\alpha$. Thus, $y = (\theta_{\beta\alpha\beta^{-1}, \beta})^{-1} \theta_{\beta, \alpha}$. This implies $y \overline{y} = 1$ and therefore $\varphi_\beta(l_\alpha) \overline{\varphi_\beta(l_\alpha)} = y \overline{y} l_{\beta\alpha\beta^{-1}} \overline{l_{\beta\alpha\beta^{-1}}} = 1$. Hence $\varphi_\beta(\overline{l_\alpha}) = \overline{\varphi_\beta(l_\alpha)}$.

If $K = \mathbf{C}$ with usual complex conjugation then $\eta(l_\alpha, \overline{l_\alpha}) = 1$ for all α so that L is unitary.

6.3. Theorem. *Let π be a group. The underlying crossed π -algebra of a Hermitian $(1+1)$ -dimensional HQFT with target $K(\pi, 1)$ has a Hermitian structure in a natural way. This establishes an equivalence between the category $HQ_2(K(\pi, 1))$ of $(1+1)$ -dimensional Hermitian HQFT's with target $K(\pi, 1)$ and the category $HQ_2(\pi)$ of Hermitian crossed π -algebras. Similar results hold in the unitary setting.*

Proof. Consider a Hermitian HQFT (A, τ) with target $K(\pi, 1)$ and its underlying crossed π -algebra L . The Hermitian structure on (A, τ) yields for each $\alpha \in \pi$ a non-degenerate Hermitian pairing $\langle \cdot, \cdot \rangle_\alpha : L_\alpha \times L_\alpha \rightarrow K$. By the non-degeneracy of η there is a unique antilinear isomorphism $b \mapsto \overline{b} : L_\alpha \rightarrow L_{\alpha^{-1}}$ such that

$$(6.3.a) \quad \langle a, b \rangle_\alpha = \eta(a, \overline{b})$$

for any $a, b \in L_\alpha$. The identity $\langle a, b \rangle_\alpha = \overline{\langle b, a \rangle_\alpha}$ may be rewritten as

$$(6.3.b) \quad \eta(a, \overline{b}) = \overline{\eta(\overline{a}, b)}$$

for any $a, b \in L$. We shall check that the homomorphism $b \mapsto \overline{b}$ defines a Hermitian structure on L . To this end we shall apply axiom (1.7.2) to the X -surfaces of types $C_{-+}(\alpha; \beta^{-1})$, $C_{--}(\alpha; 1)$ and $D_{--+}(\alpha, \beta; 1, 1)$ (cf. Section 4.6).

For $W = C_{-+}(\alpha; \beta^{-1})$, we have $-W = C_{-+}(\beta\alpha\beta^{-1}; \beta)$. By definition, $\tau(W) = \varphi_\beta : L_\alpha \rightarrow L_{\beta\alpha\beta^{-1}}$ and $\tau(-W) = \varphi_{\beta^{-1}} : L_{\beta\alpha\beta^{-1}} \rightarrow L_\alpha$. Axiom (1.7.2) yields

$$\langle \varphi_\beta(a), b \rangle_{\beta\alpha\beta^{-1}} = \langle a, \varphi_{\beta^{-1}}(b) \rangle_\alpha$$

for any $a \in L_\alpha, b \in L_{\beta\alpha\beta^{-1}}$. Substituting $c = \varphi_{\beta^{-1}}(b) \in L_\alpha$ we obtain an equivalent formula

$$\langle \varphi_\beta(a), \varphi_\beta(c) \rangle_{\beta\alpha\beta^{-1}} = \langle a, c \rangle_\alpha$$

for any $a, c \in L_\alpha$. This is equivalent to $\eta(\varphi_\beta(a), \overline{\varphi_\beta(c)}) = \eta(a, \overline{c})$. Since the form η is invariant under φ_β and non-degenerate, the latter formula is equivalent to $\varphi_\beta(\overline{c}) = \overline{\varphi_\beta(c)}$.

For $W = C_{--}(\alpha; 1)$, we have $-W = C_{++}(\alpha; 1)$. By definition, $\tau(W)$ is the pairing $a \otimes b \mapsto \eta(a, b) : L_\alpha \otimes L_{\alpha^{-1}} \rightarrow K$ where $a \in L_\alpha, b \in L_{\alpha^{-1}}$. The homomorphism $\tau(-W) : K \rightarrow L_\alpha \otimes L_{\alpha^{-1}}$ sends 1_K into a sum $\sum_i a_i \otimes b_i$ with $a_i \in L_\alpha, b_i \in L_{\alpha^{-1}}$ such that $\sum_i \eta(a, a_i) b_i = a$ for any $a \in L_{\alpha^{-1}}$ (cf. Section 5.1). Axiom (1.7.2) for W is equivalent to

$$\langle \eta(a, b), 1_K \rangle_\emptyset = \langle a \otimes b, \sum_i a_i \otimes b_i \rangle$$

where on the right-hand side we have the Hermitian form on $L_\alpha \otimes L_{\alpha^{-1}}$ induced by the Hermitian forms on the factors. By (1.7.1), $\langle \eta(a, b), 1_K \rangle_\emptyset = \eta(a, b)$. On the right-hand side we have

$$\begin{aligned} \langle a \otimes b, \sum_i a_i \otimes b_i \rangle &= \sum_i \langle a, a_i \rangle \langle b, b_i \rangle = \sum_i \eta(a, \overline{a_i}) \eta(b, \overline{b_i}) \\ &= \overline{\sum_i \eta(\overline{a}, a_i) \eta(\overline{b}, b_i)} = \overline{\eta(\overline{b}, \sum_i \eta(\overline{a}, a_i) b_i)} = \overline{\eta(\overline{b}, \overline{a})} = \overline{\eta(\overline{a}, \overline{b})}. \end{aligned}$$

This gives $\overline{\eta(a, b)} = \eta(\overline{a}, \overline{b})$. Using this formula and the non-degeneracy of η we observe that (6.3.b) is equivalent to the involutivity of $a \mapsto \overline{a} : L \rightarrow L$.

It remains to analyze the case $W = D_{--}(\alpha, \beta; 1, 1)$. By definition, $\tau(W)$ is the multiplication $a \otimes b \mapsto ab : L_\alpha \otimes L_\beta \rightarrow L_{\alpha\beta}$ where $a \in L_\alpha, b \in L_\beta$. Clearly, $-W = D_{++}(\beta, \alpha; 1, 1)$. The homomorphism $\tau(-W) : L_{\alpha\beta} \rightarrow L_\beta \otimes L_\alpha$ sends any $c \in L_{\alpha\beta}$ to a sum $\sum_j t_j \otimes z_j$ with $t_j \in L_\beta, z_j \in L_\alpha$ such that $cy = \sum_j \eta(y, t_j) z_j$ for all $y \in L_{\beta^{-1}}$. Axiom (1.7.2) for W is equivalent to

$$\langle ab, c \rangle_{\alpha\beta} = \sum_j \langle a, z_j \rangle_\alpha \langle b, t_j \rangle_\beta$$

for any $a \in L_\alpha, b \in L_\beta, c \in L_{\alpha\beta}$. The left-hand side is equal to $\eta(ab, \overline{c}) = \eta(a, b\overline{c})$ while the right-hand side is equal to

$$\sum_j \langle a, z_j \rangle_\alpha \langle b, t_j \rangle_\beta = \overline{\sum_j \eta(\overline{a}, z_j) \eta(\overline{b}, t_j)} = \overline{\eta(\overline{a}, \sum_j \eta(\overline{b}, t_j) z_j)} = \overline{\eta(\overline{a}, c\overline{b})} = \eta(a, \overline{c\overline{b}}).$$

Thus $\eta(a, b\overline{c}) = \eta(a, \overline{c\overline{b}})$ for all $a, b, c \in L$. By the non-degeneracy of η , $b\overline{c} = \overline{c\overline{b}}$. Hence $\overline{b\overline{c}} = c\overline{b}$. We conclude that $b \mapsto \overline{b}$ is a Hermitian structure on L .

Conversely, having a Hermitian structure on L we define a Hermitian structure on (A, τ) by (6.3.a) and (1.7.1). The arguments above verify axiom (1.7.2) for the X -surfaces $C_{-+}(\alpha; \beta^{-1}), C_{--}(\alpha; 1), D_{--}(\alpha, \beta; 1, 1)$. Since $C_{++} = -C_{--}$ and (1.7.2) for W is equivalent to (1.7.2) for $-W$, this axiom holds also for $W = C_{++}$. Clearly $-B_+ = B_-$. Therefore for $W = B_+$, axiom (1.7.2) is equivalent to

$$(6.3.c) \quad \langle 1_L, b \rangle_1 = \langle \tau(B_+)(1_K), b \rangle_1 = \langle 1_K, \tau(B_-)(b) \rangle_\emptyset = \overline{\tau(B_-)(\overline{b})}$$

for any $b \in L_1$. Recall (see Section 5.1) that $\tau(B_-)(b) = \eta(b, 1_L) \in K$. Therefore (6.3.c) is equivalent to $\langle 1_L, b \rangle_1 = \overline{\eta(b, 1_L)}$. Applying the conjugation in K , we obtain an equivalent formula $\langle b, 1_L \rangle_1 = \eta(b, 1_L)$. Since $\langle b, 1_L \rangle_1 = \eta(b, \overline{1_L})$, formula (6.3.c) follows from $\overline{1_L} = 1_L$.

Now, every X -surface W can be obtained by gluing from the discs with holes of types $B_+, C_{-+}(\alpha; \beta), C_{--}(\alpha; 1), C_{++}(\alpha; 1), D_{--+}(\alpha, \beta; 1, 1)$ (cf. Section 5.1). Therefore (1.7.2) holds for W . Thus, the functor $HQ_2(K(\pi, 1)) \rightarrow HQ_2(\pi)$ is surjective on objects. That this functor is an equivalence of categories is straightforward (cf. Section 5.1).

Part III. Lattice models for $(1+1)$ -dimensional HQFT's

7. Biangular π -algebras and lattice HQFT's

By a lattice topological quantum field theory one means a TQFT which can be computed in terms of partition functions (or state sums) on triangulations or cellular decompositions of manifolds. Lattice TQFT's are well known in dimensions 2 and 3, see [BP], [FHK], [TV]. In this section we introduce a lattice HQFT in dimension 2.

7.1. Biangular π -algebras. Let $L = \bigoplus_{\alpha \in \pi} L_\alpha$ be a π -algebra. Given $\ell \in L$, denote by μ_ℓ the left multiplication by ℓ sending any $a \in L$ into $\ell a \in L$. Clearly, if $\ell \in L_1$ then $\mu_\ell(L_\alpha) \subset L_\alpha$ for all $\alpha \in \pi$. We call L a *biangular π -algebra* if it satisfies the following two conditions:

(i) for any $\ell \in L_1$ and any $\alpha \in \pi$,

$$(7.1.a) \quad \text{Tr}(\mu_\ell|_{L_\alpha} : L_\alpha \rightarrow L_\alpha) = \text{Tr}(\mu_\ell|_{L_1} : L_1 \rightarrow L_1),$$

(ii) for any $\alpha \in \pi$, the bilinear form $\eta : L_\alpha \otimes L_{\alpha^{-1}} \rightarrow K$ defined by

$$(7.1.b) \quad \eta(a, b) = \text{Tr}(\mu_{ab}|_{L_1} : L_1 \rightarrow L_1)$$

(where $a \in L_\alpha, b \in L_{\alpha^{-1}}$) is non-degenerate.

Condition (i) implies that

$$\text{Dim}(L_\alpha) = \text{Tr}(\text{id} : L_\alpha \rightarrow L_\alpha) = \text{Tr}(\mu_1|_{L_\alpha} : L_\alpha \rightarrow L_\alpha) = \text{Tr}(\mu_1|_{L_1} : L_1 \rightarrow L_1)$$

does not depend on $\alpha \in \pi$. This shows in particular that the crossed π -algebras are not necessarily biangular.

The bilinear form η on a biangular π -algebra L defined by (7.1.b) is symmetric: if $a \in L_\alpha, b \in L_{\alpha^{-1}}$ then

$$\begin{aligned} \eta(b, a) &= \text{Tr}(\mu_{ba}|_{L_1}) = \text{Tr}(\mu_b \mu_a|_{L_1} : L_1 \rightarrow L_1) = \text{Tr}(\mu_a \mu_b|_{L_\alpha} : L_\alpha \rightarrow L_\alpha) \\ &= \text{Tr}(\mu_{ab}|_{L_\alpha} : L_\alpha \rightarrow L_\alpha) = \text{Tr}(\mu_{ab}|_{L_1} : L_1 \rightarrow L_1) = \eta(a, b). \end{aligned}$$

It is clear that the inner product η makes L a Frobenius π -algebra.

The group ring $K[\pi]$ with canonical π -algebra structure is biangular. More generally, the π -algebra constructed in Section 3.3 is biangular. It follows from definitions and the standard properties of the trace that the direct sums and tensor products of biangular π -algebras are biangular π -algebras. More examples of biangular algebras can be obtained using the pull-back and push-forward constructions, see Section 3.1. Namely, the pull-back of a biangular π -algebra along any group homomorphism $q : \pi' \rightarrow \pi$ is a biangular π' -algebra. If q is a surjective homomorphism with finite kernel and the order of $\text{Ker } q$ is invertible in K , then the push-forward along q of a biangular π' -algebra is a biangular π -algebra. Further examples of biangular algebras will be given in Section 10.

Our interest in biangular π -algebras is due to the fact that each such algebra L gives rise to a $(1+1)$ -dimensional HQFT with target $K(\pi, 1)$. The underlying crossed π -algebra of this HQFT is called the π -center of L . (This term is justified by the examples given in Section 10.2). We give here a direct algebraic description of the π -center of L .

Observe first that for every $\alpha \in \pi$, the non-degenerate form $\eta : L_\alpha \otimes L_{\alpha^{-1}} \rightarrow K$ yields a canonical element $b_\alpha \in L_\alpha \otimes L_{\alpha^{-1}}$ (cf. Lemma 7.1.1 below). We expand

$$(7.1.c) \quad b_\alpha = \sum_i p_i^\alpha \otimes q_i^\alpha$$

where i runs over a finite set of indices and $p_i^\alpha \in L_\alpha, q_i^\alpha \in L_{\alpha^{-1}}$. Since η is symmetric, the vector $b_{\alpha^{-1}}$ is obtained from b_α by permutation of the tensor factors. The element b_α is characterized by the following property: for any $p \in L_\alpha$,

$$p = \sum_i \eta(p, q_i^\alpha) p_i^\alpha.$$

Note that the sum $\sum_i p_i^\alpha q_i^\alpha \in L_1$ does not depend on the choice of expansion (7.1.c). Below we shall prove that for all $\alpha \in \pi$,

$$(7.1.d) \quad \sum_i p_i^\alpha q_i^\alpha = 1_L.$$

Define a K -homomorphism $\psi_\alpha : L \rightarrow L$ by $\psi_\alpha(a) = \sum_i p_i^\alpha a q_i^\alpha$ where $a \in L$. This homomorphism does not depend on the choice of expansion (7.1.c) and sends each $L_\beta \subset L$ into $L_{\alpha\beta\alpha^{-1}}$. It turns out that $\psi_\alpha \psi_{\alpha'} = \psi_{\alpha\alpha'}$ for any $\alpha, \alpha' \in \pi$ and ψ_α is adjoint to $\psi_{\alpha^{-1}}$ with respect to η . In particular, ψ_1 is an orthogonal projection onto a submodule $C = \bigoplus_{\alpha \in \pi} C_\alpha$ of L where $C_\alpha = \psi_1(L_\alpha) \subset L_\alpha$. Formula (7.1.d) implies that $1_L \in C$. It turns out that C is a subalgebra of L . Moreover, the pairing $\eta|_C : C \otimes C \rightarrow K$ and the homomorphism $\pi \rightarrow \text{Aut } C$ sending each $\alpha \in \pi$ to $\psi_\alpha|_C : C \rightarrow C$ make C a crossed π -algebra. The crossed π -algebra C is the π -center of L . All these claims follow directly by applying the definitions of Section 4 to the HQFT associated with L (cf. Section 8.6).

It is easy to check that if $L = K[\pi]$ or more generally if L is the π -algebra constructed in Section 3.3, then $\psi_1 = \text{id}_L$ so that $C^L = L$.

We finish this subsection with a proof of the existence of b_α , check (7.1.d), and state a lemma which will be used in Section 8. In the remaining part of Section 7 we give a construction of the lattice HQFT associated with a biangular π -algebra L . A dual construction of the same HQFT is given in Section 8.

7.1.1. Lemma. *Let P, Q be projective K -modules of finite type and let $\eta : P \otimes Q \rightarrow K$ be a non-degenerate bilinear form. There is a unique element $b \in P \otimes Q$ such that for any expansion $b = \sum_i p_i \otimes q_i$ into a finite sum with $p_i \in P, q_i \in Q$ and any $p \in P, q \in Q$, we have*

$$p = \sum_i \eta(p, q_i) p_i, \quad q = \sum_i \eta(p_i, q) q_i.$$

Moreover, for any K -homomorphism $f : P \rightarrow P$, we have $\text{Tr}(f) = \sum_i \eta(f(p_i), q_i)$.

Proof. Denote by ρ the homomorphism $Q \rightarrow P^*$ defined by $\rho(q)(p) = \eta(p, q)$ for any $p \in P, q \in Q$. Since η is non-degenerate, ρ is an isomorphism.

To any element $b = \sum_i p_i \otimes q_i$ of $P \otimes Q$ we associate a homomorphism $\nu_b : P^* \rightarrow Q$ sending each $x \in P^*$ into $\nu_b(x) = \sum_i x(p_i) q_i \in Q$. Since P and Q are projective modules, the formula $b \mapsto \nu_b$ defines an isomorphism $P \otimes Q = \text{Hom}(P^*, Q)$.

Observe that the identity $p = \sum_i \eta(p, q_i) p_i$ for all $p \in P$ is equivalent to $\rho \nu_b = \text{id}_{P^*}$. Indeed, $\rho \nu_b$ sends any $x \in P^*$ into the homomorphism $P \rightarrow K$ which maps each $p \in P$ into $\eta(p, \sum_i x(p_i) q_i) = x(\sum_i \eta(p, q_i) p_i)$. The equality $\rho \nu_b(x) = x$ holds for all $x \in P^*$ if and only if $\sum_i \eta(p, q_i) p_i = p$ for all $p \in P$. A similar argument shows that the identity $q = \sum_i \eta(p_i, q) q_i$ for all $q \in Q$ is equivalent to $\nu_b \rho = \text{id}_Q$. Thus, the only element b satisfying conditions of the lemma is determined from $\nu_b = \rho^{-1} \in \text{Hom}(P^*, Q)$.

The second claim of the lemma is standard (see for instance [Tu, Lemma II.4.3.1]).

7.1.2. Proof of (7.1.d). Using the expression for the trace given in Lemma 7.1.1, we obtain for any $\ell \in L_1, \alpha \in \pi$,

$$\eta(\ell, 1_L) = \text{Tr}(\mu_\ell|_{L_1}) = \text{Tr}(\mu_\ell|_{L_\alpha} : L_\alpha \rightarrow L_\alpha) = \sum_i \eta(\ell p_i^\alpha, q_i^\alpha) = \eta(\ell, \sum_i p_i^\alpha q_i^\alpha).$$

Since the form $\eta : L_1 \otimes L_1 \rightarrow K$ is non-degenerate, $\sum_i p_i^\alpha q_i^\alpha = 1_L$.

7.1.3. Lemma. *For any $\alpha \in \pi, a \in L_{\alpha^{-1}}, b \in L_\alpha$ and any expansion (7.1.c) of b_α , we have $\sum_i \eta(a p_i^\alpha, 1_L) \eta(q_i^\alpha b, 1_L) = \eta(ab, 1_L)$.*

Proof.

$$\sum_i \eta(a p_i^\alpha, 1_L) \eta(q_i^\alpha b, 1_L) = \sum_i \eta(a, p_i^\alpha) \eta(q_i^\alpha, b) = \sum_i \eta(a, p_i^\alpha) \eta(b, q_i^\alpha)$$

$$= \eta(a, \sum_i \eta(b, q_i^\alpha) p_i^\alpha) = \eta(a, b) = \eta(ab, 1_L).$$

7.2. Maps to $K(\pi, 1)$ as combinatorial systems. It is well known that the homotopy classes of maps from a CW-complex to the Eilenberg-MacLane space $X = K(\pi, 1)$ admit a combinatorial description in terms of elements of π assigned to the 1-cells. These elements may be viewed as the holonomies of the corresponding principal π -bundles. Here we recall this description adapting it to our setting.

Let T be a CW-complex with underlying topological space \tilde{T} . By vertices, edges and faces of T we mean 0-cells, 1-cells and 2-cells of T , respectively. Denote the set of vertices of T by $\text{Vert}(T)$. Each oriented edge e of T leads from an initial vertex, $i_e \in \text{Vert}(T)$, to a terminal vertex, $t_e \in \text{Vert}(T)$ (they may coincide). Denote by $\text{Edg}(T)$ the set of oriented edges of T . Inverting the orientation we obtain an involution $e \mapsto e^{-1}$ on $\text{Edg}(T)$.

Each face Δ of T is obtained by adjoining a 2-disc to the 1-skeleton T^1 of T along a map $f_\Delta : S^1 \rightarrow T^1$. In general this map may be rather wild; we shall require the following property of regularity. We say that the CW-complex T is *regular* if for any face Δ of T the pre-image $f_\Delta^{-1}(\text{Vert}(T)) \subset S^1$ of $\text{Vert}(T)$ is a finite non-empty set which splits the circle S^1 into arcs mapped by f_Δ homeomorphically onto certain edges of T . These arcs in S^1 are called the *sides* of Δ . The image of each side of Δ under f_Δ is an edge of T . An orientation of S^1 determines an orientation and a cyclic order of the sides of Δ . The corresponding cyclically ordered oriented edges of T form the boundary of Δ . Examples of regular CW-complexes are provided by triangulated spaces.

Assume that T is regular. Assume also that a certain subset (possibly empty) of $\text{Vert}(T)$ is distinguished; the vertices belonging to this subset are called the *base vertices* of T . By a π -system $\{g_e\}$ on T we mean a function which assigns to every $e \in \text{Edg}(T)$ an element $g_e \in \pi$ such that

- (i) for any $e \in \text{Edg}(T)$ we have $g_{e^{-1}} = (g_e)^{-1}$;
- (ii) if ordered oriented edges e_1, e_2, \dots, e_n of T with $n \geq 1$ form the boundary of a face then $g_{e_1} g_{e_2} \dots g_{e_n} = 1$.

Let F_T be the set of maps $\text{Vert}(T) \rightarrow \pi$ taking value $1 \in \pi$ on all base vertices of T . The set F_T is a group with pointwise multiplication. If $g = \{g_e\}$ is a π -system on T and $\gamma \in F_T$ then the formula $(\gamma g)_e = \gamma(i_e) g_e (\gamma(t_e))^{-1}$ yields a new π -system γg on T . This defines a left action of F_T on the set of π -systems on T . We say that two π -systems g, g' on T are *homotopic* if they lie in the same orbit of this action, i.e., if there is $\gamma \in F_T$ such that $g' = \gamma g$. It is clear that homotopy is an equivalence relation.

Every π -system g on T gives rise to a map $\tilde{g} : \tilde{T} \rightarrow X = K(\pi, 1)$ which sends all vertices of T into the base point of X and sends each oriented edge $e \in \text{Edg}(T)$ into a loop in X representing $g_e \in \pi$. An elementary obstruction theory shows that the formula $g \mapsto \tilde{g}$ establishes a bijective correspondence between the homotopy classes of π -systems on T and the pointed homotopy classes of maps $\tilde{T} \rightarrow X$. (These are classes of maps sending all the base vertices of T in the base point of

X modulo homotopies constant on the base vertices).

By a CW-decomposition of an X -cobordism (or an X -manifold) W we shall mean a regular CW-decomposition such that ∂W is a subcomplex and all the base points are among vertices. A π -system g on a CW-decomposition T of W is said to be *characteristic* if the map $\tilde{g} : W = \tilde{T} \rightarrow X$ is homotopic (in the pointed category) to the given characteristic map $W \rightarrow X$. Note that the characteristic π -systems on T form a single homotopy class of π -systems.

7.3. State sums on closed X -surfaces. Fix a biangular π -algebra L . Let W be a closed X -surface, i.e., a closed oriented surface endowed with a map $W \rightarrow X = K(\pi, 1)$. (We do not provide closed surfaces with base points, cf. Section 1.1). Here we define a state sum invariant $\tau(W) = \tau_L(W) \in K$.

Choose a (regular) CW-decomposition T of W . By a *flag* in T we mean a pair (a face Δ of T , a side e of Δ). The flag (Δ, e) determines an orientation of e such that Δ lies on the right of e . This means that the pair (a vector looking from a point of e inside Δ , the oriented side e) is positively oriented with respect to the given orientation of W . By abuse of notation, we shall denote the edge of T corresponding to e by the same letter e .

Let g be a characteristic π -system on T . With each flag (Δ, e) in T we associate the K -module $L(\Delta, e, g) = L_{g_e}$ where the orientation of e is determined by Δ as above. Each unoriented edge e of T appears in two flags, $(\Delta, e), (\Delta', e)$, and inherits from them opposite orientations. Since the corresponding values of g are inverse to each other, we have the canonical vector $b_{g_e} \in L(\Delta, e, g) \otimes L(\Delta', e, g)$ introduced in Section 7.1. The tensor product of these vectors over all unoriented edges of T is an element, $B_g \in \otimes_{(\Delta, e)} L(\Delta, e, g)$ where (Δ, e) runs over all flags in T .

Let Δ be a face of T with $n \geq 1$ oriented ordered sides e_1, \dots, e_n whose orientation is determined by Δ as above and the order is chosen so that the terminal endpoint of e_r is the initial endpoint of e_{r+1} for $r = 1, \dots, n \pmod{n}$. Note that $g_{e_1} \dots g_{e_n} = 1$. The n -linear form

$$L(\Delta, e_1, g) \otimes L(\Delta, e_2, g) \otimes \dots \otimes L(\Delta, e_n, g) \rightarrow K$$

defined by $(a_1, a_2, \dots, a_n) \mapsto \eta(a_1 a_2 \dots a_n, 1_L)$ with $a_r \in L(\Delta, e_r, g)$ for $r = 1, \dots, n$ is invariant under cyclic permutations and therefore is determined by Δ . The tensor product of these multilinear forms over all faces of T is a homomorphism, $D_g : \otimes_{(\Delta, e)} L(\Delta, e, g) \rightarrow K$ where (Δ, e) runs over all flags in T . Set $\langle g \rangle = D_g(B_g) \in K$.

7.3.1. Claim. *The state sum $\langle g \rangle$ does not depend on the choice of g in its homotopy class and does not depend on the choice of T .*

For a proof, see Section 8.

Claim 7.3.1 implies that $\tau(W) = \langle g \rangle = D_g(B_g) \in K$ is a well defined invariant of the X -surface W . By the very definition, $\tau(W)$ depends only on the homotopy class of the characteristic map $W \rightarrow X$ and is multiplicative with respect to disjoint union of closed X -surfaces.

For connected W , we can give an explicit formula for $\tau(W)$. We need the following notation. For $b \in L_\alpha \otimes L_{\alpha^{-1}}, b' \in L_\beta \otimes L_{\beta^{-1}}$ with $\alpha, \beta \in \pi$, we define $[b, b'] \in L_{\alpha\beta\alpha^{-1}\beta^{-1}}$ as follows: expand b, b' into finite sums $b = \sum_i p_i \otimes q_i, b' = \sum_j p'_j \otimes q'_j$ with $p_i \in L_\alpha, q_i \in L_{\alpha^{-1}}, p'_j \in L_\beta, q'_j \in L_{\beta^{-1}}$ and set

$$[b, b'] = \sum_{i,j} p_i p'_j q_i q'_j \in L_{\alpha\beta\alpha^{-1}\beta^{-1}}.$$

It is clear that $[b, b']$ does not depend on the choice of the expansions of b, b' . Assume now that W is a closed connected oriented X -surface of genus $n \geq 1$. Consider the CW-decomposition of W formed by one vertex, $2n$ oriented loops a_1, a_2, \dots, a_{2n} and one face with boundary $\prod_{r=1}^n (a_{2r-1} a_{2r} a_{2r-1}^{-1} a_{2r}^{-1})$. We choose this CW-decomposition so that the intersection number $[a_1] \cdot [a_2]$ of the integer homology classes represented by the loops a_1, a_2 equals -1 . We deform the characteristic map $g : W \rightarrow X$ so that it sends the only vertex of W into the base point of X . Consider the induced homomorphism $g_\#$ of the fundamental groups and set $\alpha_s = g_\#(a_s) \in \pi$ for $s = 1, \dots, 2n$. Recall the canonical element $b_{\alpha_s} \in L_{\alpha_s} \otimes L_{\alpha_s^{-1}}$. Computing $\tau(W)$ from this CW-decomposition of W , we obtain

$$\tau(W) = \eta\left(\prod_{r=1}^n [b_{\alpha_{2r-1}}, b_{\alpha_{2r}}], 1_L\right)$$

where $\prod_{r=1}^n [b_{\alpha_{2r-1}}, b_{\alpha_{2r}}] \in L_1$. In the case $W = S^2$, all maps $W \rightarrow X$ are homotopic to the constant map. The invariant $\tau(S^2)$ can be computed from the CW-decomposition of S^2 consisting of one vertex, one loop and two faces (hemispheres). Fix an expansion of the canonical element $b_1 = \sum_i p_i \otimes q_i \in L_1 \otimes L_1$. We have

$$\tau(S^2) = \sum_i \eta(p_i, 1_L) \eta(q_i, 1_L) = \eta\left(\sum_i \eta(q_i, 1_L) p_i, 1_L\right) = \eta(1_L, 1_L) = \text{Dim} L_1.$$

7.4. A lattice $(1+1)$ -dimensional HQFT. We extend the invariant τ of closed X -surfaces constructed above to a $(1+1)$ -dimensional HQFT with target $X = K(\pi, 1)$. The construction goes in three steps. At the first step we assign K -modules to so-called trivialized 1-dimensional X -manifolds. At the second step we extend this assignment to a preliminary $(1+1)$ -dimensional HQFT defined for trivialized X -manifolds and X -cobordisms with trivialized boundary. At the third step we get rid of the trivializations.

Step 1. We say that a 1-dimensional X -manifold M is *trivialized* if it is provided with a CW-decomposition T (such that the base points of the components of M are among the vertices) and a characteristic π -system g on T . We provide each edge e of M with canonical orientation induced by the one of M . We associate with the trivialized 1-dimensional X -manifold $M = (M, T, g)$ the

K -module $A(M) = \otimes_e L_{g_e}$ where e runs over all canonically oriented edges of T . It is clear that the module $A(M)$ is projective of finite type. For disjoint trivialized 1-dimensional X -manifolds M, N , we have $A(M \amalg N) = A(M) \otimes_K A(N)$. If $M = \emptyset$ then by definition M is trivialized and $A(M) = K$.

Step 2. Consider a 2-dimensional X -cobordism (W, M_0, M_1) as defined in Section 1.1. Assume that the bases M_0, M_1 of W are trivialized. We define a K -homomorphism $\tau(W) : A(M_0) \rightarrow A(M_1)$ as follows. Choose a regular CW-decomposition T of W extending the given CW-decomposition of $\partial W = (-M_0) \cup M_1$. Choose a characteristic π -system g on T extending the given π -system on ∂W . Applying the constructions of Section 7.3 to g we obtain a vector $B_g \in \otimes_{(\Delta, e)} L(\Delta, e, g)$ where (Δ, e) runs over flags in T such that e does not lie on ∂W . As in Section 7.3, we obtain a homomorphism $D_g : \otimes_{(\Delta, e)} L(\Delta, e, g) \rightarrow K$ where (Δ, e) runs over all flags in T . Contracting these two tensors we obtain a homomorphism $\langle g \rangle : \otimes_{(\Delta, e \subset \partial W)} L(\Delta, e, g) \rightarrow K$. We provide each edge e lying on M_r ($r = 0, 1$) with the orientation induced by the one of M_r . This orientation of e coincides with the one induced by the only face attached to e if $e \subset M_0$ and is opposite to it if $e \subset M_1$. Therefore $L(\Delta, e, g) = L_{g_e}$ if $e \subset M_0$ and $L(\Delta, e, g) = L_{(g_e)^{-1}}$ if $e \subset M_1$. We identify $L_{(g_e)^{-1}}$ with the dual of L_{g_e} using the inner product η . In this way we can view $\langle g \rangle$ as a homomorphism

$$A(M_0) = \bigotimes_{e \subset M_0} L_{g_e} \rightarrow \bigotimes_{e \subset M_1} L_{g_e} = A(M_1).$$

We claim that $\langle g \rangle$ does not depend on the choice of g and T ; this will be checked in Section 8. Set $\tau(W) = \langle g \rangle : A(M_0) \rightarrow A(M_1)$. By the very definition, $\tau(W)$ depends only on the homotopy class of the characteristic map $W \rightarrow X$ and satisfies axiom (1.2.5). The next lemma describes the behavior of τ under the gluing of X -cobordisms.

7.4.1. Lemma. *Let M_0, M_1, N be trivialized 1-dimensional X -manifolds (possibly void). If a 2-dimensional X -cobordism (W, M_0, M_1) is obtained from two 2-dimensional X -cobordisms (W_0, M_0, N) and (W_1, N, M_1) by gluing along the identity map $N = N$ then $\tau(W) = \tau(W_1) \circ \tau(W_0) : A(M_0) \rightarrow A(M_1)$.*

Proof. For $r = 0, 1$, fix a regular CW-decomposition T_r of W_r extending the given CW-decomposition of the boundary. Gluing T_0 and T_1 along N we obtain a regular CW-decomposition, T , of W . The lemma is a reformulation of the following claim:

(*) *Let for $r = 0, 1$, g_r be a π -system on T_r extending the given π -systems on M_r and N . Let g be the unique π -system on T extending g_0 and g_1 . Then the homomorphism $\langle g \rangle : A(M_0) \rightarrow A(M_1)$ is the composition of $\langle g_0 \rangle : A(M_0) \rightarrow A(N)$ and $\langle g_1 \rangle : A(N) \rightarrow A(M_1)$.*

This claim directly follows from the definition of $\langle g \rangle$.

Step 3. The constructions above can be summarized as follows. To any trivialized 1-dimensional X -manifold M we assign a module $A(M)$. This module is natural with respect to X -homeomorphisms preserving the trivializations. To any X -cobordism (W, M_0, M_1) between trivialized 1-dimensional X -manifolds we assign a homomorphism $\tau(W) : A(M_0) \rightarrow A(M_1)$. This data looks like a HQFT and satisfies the natural versions of the axioms (1.2.1) - (1.2.6) and (1.2.8). However, in general $\tau(M \times [0, 1]) \neq \text{id}_{A(M)}$. There is a standard procedure which allows to pass from such a pseudo-HQFT to a genuine $(1+1)$ -dimensional HQFT. This procedure is described in detail in a similar setting in [Tu, Section VII.3] cf. also Section 2.1. The idea is that if t_1, t_2 are two trivializations of a 1-dimensional X -manifold M then the cylinder $W = M \times [0, 1]$ (mapped to X via the composition of the projection $M \times [0, 1] \rightarrow M$ with the characteristic map $M \rightarrow X$) is an X -cobordism between (M, t_1) and (M, t_2) . This gives a homomorphism $p(t_1, t_2) = \tau(W) : A(M, t_1) \rightarrow A(M, t_2)$. By Lemma 7.4.1, we have the identity $p(t_1, t_3) = p(t_2, t_3)p(t_1, t_2)$. Taking $t_1 = t_2 = t_3$ we obtain that $p(t_1, t_1)$ is a projection onto a direct summand, $\hat{A}(M, t_1)$, of $A(M, t_1)$. Moreover, $p(t_1, t_2)$ maps $\hat{A}(M, t_1)$ isomorphically onto $\hat{A}(M, t_2)$. This allows us to identify the modules $\{\hat{A}(M, t)\}_t$ where t runs over all trivializations of M along these canonical isomorphisms and to obtain a module, $\hat{A}(M)$, independent of t . Next we observe that for any 2-dimensional X -cobordism (W, M_0, M_1) with trivialized bases, the homomorphism $\tau(W) : A(M_0) \rightarrow A(M_1)$ maps $\hat{A}(M_0) \subset A(M_0)$ into $\hat{A}(M_1) \subset A(M_1)$. This yields a homomorphism $\hat{\tau}(W) : \hat{A}(M_0) \rightarrow \hat{A}(M_1)$ independent of the trivializations of the bases. The modules \hat{A} and the homomorphisms $\hat{\tau}$ form the $(1+1)$ -dimensional HQFT with target X associated with L . Note that if W is a closed X -surface, then $\hat{\tau}(W) = \tau(W)$.

7.5. Remark. The HQFT constructed above is additive (resp. multiplicative) with respect to direct sums (resp. tensor products) of biangular π -algebras. This HQFT is functorial with respect to pull-backs of biangular group-algebras along group homomorphisms. Note that any group homomorphism $q : \pi' \rightarrow \pi$ induces a map $X' = K(\pi', 1) \rightarrow K(\pi, 1) = X$ denoted f_q . It is easy to deduce from definitions that if a biangular π' -algebra $L' = q^*(L)$ is the pull-back of a biangular π -algebra L along q , then the lattice HQFT determined by L' is obtained from the lattice HQFT determined by L via pulling back along f_q , cf. Section 2.2. In particular, each closed X' -surface $W = (W, g : W \rightarrow X')$ gives rise to a closed X -surface $W_q = (W, f_q g : W \rightarrow X)$ and $\tau_{L'}(W) = \tau_L(W_q)$. For instance, consider the homomorphism $q : \pi \rightarrow \{1\}$ where $\{1\}$ is the trivial group. If L is the 1-dimensional $\{1\}$ -algebra K , then $q^*(L)$ is the group ring $K[\pi]$ and $\tau_{K[\pi]}(W) = \tau_L(W_q) = 1 \in K$.

The relations with push-forwards of biangular algebras are more subtle, cf. Remark 9.7.

8. Lattice models on skeletons

8.1. Skeletons of 2-manifolds. We shall use the language of graphs. By a

graph we mean a 1-dimensional CW-complex. The *valency* of a vertex of a graph is the number of edges incident to this vertex (counted with multiplicity).

Let W be a compact oriented surface with based (possibly void) boundary. A *skeleton* of W is a finite graph Γ embedded in W such that

- (i) Γ has no isolated vertices (i.e., vertices of valency 0);
- (ii) each connected component of $W \setminus \Gamma$ is either an open 2-disc or a half-open 2-disc homeomorphic to $[0, 1) \times (0, 1)$ and meeting ∂W along the arc $0 \times (0, 1)$.
- (iii) each point of the set $\Gamma \cap \partial W$ is a 1-valent vertex of Γ ; the base points of ∂W do not belong to Γ .

The 1-valent vertices of Γ lying on ∂W are called *feet* of Γ . An open subset of Γ consisting of a foot and the unique open edge incident to it is called a *leg* of Γ .

Conditions (i) - (iii) imply that each component of ∂W contains at least one foot of Γ and that the part of Γ lying in a connected component of W is connected. Note that a skeleton may have loops (i.e., edges with coinciding endpoints) and multiple edges (i.e., different edges with the same endpoints). It may also have 2-valent vertices and 1-valent vertices lying in $\text{Int}(W)$.

There are four basic local moves on the skeletons of W . These moves transform a skeleton of W into another skeleton of W with the same feet. The first move, called the *contraction* move, contracts an edge e into a point provided e is neither a loop nor a leg. The second move, called the *biangular* move, introduces a small biangle (or bigon) in the middle of an edge of the skeleton. More precisely, this move replaces a small subarc of the edge with two parallel embedded subarcs with the same endpoints disjoint from the rest of the skeleton. The number of vertices decreases by 1 under the contraction move and increases by 2 under the biangular move. The third and forth moves are the inverses of the contraction and biangular moves.

Another useful move on a skeleton Γ adds a small loop based at an edge of Γ . The loop should be disjoint from Γ except at its endpoint and should bound a small disc in $W \setminus \Gamma$. This *loop move* is a composition of a biangular move and a contraction along one of the sides of the biangle. Conversely, the biangular move is a composition of a loop move and an inverse contraction move.

8.1.1. Lemma. *Any two skeletons of W having the same number of feet on every component of ∂W can be related by a finite sequence of basic moves and ambient isotopy of W constant on the base points of ∂W .*

Proof. This lemma is a simple application of the known description of the moves relating generalized spines of W . By a *generalized spine* of W we mean a finite trivalent graph G embedded in $\text{Int}(W)$ such that all connected components of $W \setminus G$ are either open 2-discs or half-open annuli $S \times [0, 1)$ where $S = S \times 0$ runs over components of ∂W . There are three local moves on the generalized spines. The first move, called the *IH*-move, replaces a piece of the spine looking like the letter *I* by a piece looking like the letter *H*. The second and third moves are the biangular move considered above and its inverse. It is well known that any two generalized spines of W can be related by a finite sequence of such moves and

ambient isotopy.

Let us call a skeleton Γ of W *trivalent* if all its vertices lying in $\text{Int}(W)$ are trivalent and at least one of these vertices is not incident to a leg. It is easy to see that any skeleton of W can be transformed by the basic moves into a trivalent skeleton. Removing from a trivalent skeleton of W all its legs we obtain a generalized spine, G , of W . We can reconstruct the skeleton by adjoining to G several legs lying in the annuli components of $W \setminus G$. Note that different ways to adjoin the legs are related by *IH*-moves. Now the results mentioned above imply that any two trivalent skeletons of W can be related by a sequence of *IH*-moves, biangular moves and inverse biangular moves. It remains to observe that each *IH*-move is a composition of a contraction move and an inverse contraction move.

8.2. Skeletons versus CW-decompositions. Let W be a compact oriented surface with based (possibly void) boundary. A regular CW-decomposition T of W such that ∂W is a subcomplex and the base points of ∂W are among the vertices yields a “dual” skeleton Γ_T of W as follows. Choose inside every 2-cell and every 1-cell of T a point called its center. Let us connect the center of each 2-cell Δ to the centers of its sides by embedded arcs lying in Δ and disjoint except at their common endpoint. A 1-cell e of T not lying in ∂W gives rise to a “dual edge” formed by two arcs joining the center of e with the centers of 2-cells adjacent to e . A 1-cell e of T lying in ∂W gives rise to a “dual edge” which is the arc joining the center of e to the center of the only 2-cell adjacent to e . The vertices of Γ_T are the centers of the 2-cells of T and the centers of the 1-cells of T lying in ∂W . The edges of Γ_T are the dual edges of the 1-cells of T . Note that the legs of Γ_T are the dual edges of the 1-cells of T contained in ∂W .

The formula $T \mapsto \Gamma_T$ establishes a bijective correspondence between the (isotopy classes of) regular CW-decompositions as above and skeletons of W . Here by isotopy we mean ambient isotopy constant on the base points. The inverse construction of $T = T_\Gamma$ from a skeleton Γ of W is as follows. Choose in each component U of $W \setminus \Gamma$ a point called its center. We assume that if U meets ∂W along an arc then the center of U is chosen on this arc. Moreover, if U meets ∂W along an arc containing a base point of W then we take this point as the center of U . The centers of the components of $W \setminus \Gamma$ are the vertices of T . Each edge e of Γ gives rise to a 1-cell e^* of T which crosses e transversally in one point and connects the centers of the components of $W \setminus \Gamma$ adjacent to e . Moreover, if e is a leg of Γ then $e^* \subset \partial W$ and $e^* \cap e$ is the foot of e . This completes the description of the 0-cells and 1-cells of T , the connected components of their complement in W are the 2-cells of T . The 2-cells of T bijectively correspond to those vertices of Γ which are not feet.

Observe that if an edge e of Γ is oriented then the dual 1-cell e^* of T_Γ can be oriented so that it crosses e from right to left, the right and left of e being determined by the orientations of W and e , cf. Section 7.3. Obviously, $(e^{-1})^* = (e^*)^{-1}$. This establishes a bijection $\text{Edg}(\Gamma) = \text{Edg}(T_\Gamma)$ equivariant with respect to the involution $e \mapsto e^{-1}$.

8.3. π -systems on skeletons. Let Γ be a skeleton of a compact oriented surface W with based (possibly void) boundary. A π -system $\{g_e\}$ on Γ is a function which assigns to any oriented edge $e \in \text{Edg}(\Gamma)$ an element $g_e \in \pi$ such that

- (i) for any $e \in \text{Edg}(\Gamma)$, we have $g_{e^{-1}} = (g_e)^{-1}$;
- (ii) for any vertex u of Γ of valency $n \geq 1$, the oriented edges e_1, e_2, \dots, e_n of Γ with terminal vertex u and the cyclic order opposite to the one induced by the orientation of W in a neighborhood of u satisfy $g_{e_1} g_{e_2} \dots g_{e_n} = 1$.

Two π -systems g, g' on Γ are said to be *homotopic* if there is a function $\gamma : \pi_0(W \setminus \Gamma) \rightarrow \pi$ such that $\gamma(U) = 1$ if U is a component of $W \setminus \Gamma$ containing a base point of ∂W , and for any $e \in \text{Edg}(\Gamma)$, we have $g'_e = \gamma(U) g_e (\gamma(V))^{-1}$ where U, V are the components of $W \setminus \Gamma$ lying on the right and left of e , respectively (possibly, $U = V$). If γ takes the value $1 \in \pi$ on all components of $W \setminus \Gamma$ except one component U then we say that g' is obtained from g by a *homotopy move* at U . It is clear that two π -systems on Γ are homotopic if and only if they can be related by homotopy moves.

Setting $g_{e^*} = g_e$ we obtain a bijective correspondence between the π -systems on Γ and on the dual CW-decomposition T_Γ . This correspondence preserves the relation of homotopy.

Every π -system $\{g_e\}$ on Γ gives rise to a map $\tilde{g} : W \rightarrow X$ which sends all vertices of T_Γ into the base point of $K(\pi, 1)$ and for each $e \in \text{Edg}(\Gamma)$ sends the oriented 1-cell e^* into a loop representing g_e . Condition (ii) above ensures that this mapping of the 1-skeleton of T_Γ extends to W . This establishes a bijective correspondence between the homotopy classes of π -systems on Γ and the set of (pointed) homotopy classes of maps $W \rightarrow X$.

Each basic move $\Gamma \mapsto \Gamma'$ on skeletons lifts to π -systems by transforming any π -system g on Γ into a π -system g' on Γ' coinciding with g on the common part of Γ and Γ' . The system g' is uniquely determined by g in the case of the contraction move, the inverse contraction move and the inverse biangular move. Under the biangular move, such a π -system g' exists and is unique up to homotopy moves at the small biangle component of $W \setminus \Gamma'$ created by the move. For all basic moves, the mapping $g \mapsto g'$ establishes a bijection between the homotopy classes of π -systems on Γ and Γ' . The classes corresponding to each other under this bijection determine the same homotopy class of maps $W \rightarrow X$.

8.4. State sums on closed X -surfaces via skeletons. Fix a biangular π -algebra L . Let W be a closed X -surface. Here we define a numerical invariant $\tau(W) = \tau_L(W) \in K$ using the skeletons of W .

Take a skeleton Γ of W and represent the characteristic map $W \rightarrow X$ by a π -system $g = \{g_e\}_e$ on Γ . Every unoriented edge of Γ gives rise to two oriented edges e, e^{-1} and to the vector $b_{g_e} \in L_{g_e} \otimes L_{(g_e)^{-1}}$. The tensor product of these vectors over all unoriented edges of Γ is an element, $B_g \in \otimes_{e \in \text{Edg}(\Gamma)} L_{g_e}$.

Consider a vertex u of Γ of valency $n \geq 1$ and n oriented edges e_1, \dots, e_n of Γ incident to u and directed towards u . We choose the cyclic order e_1, \dots, e_n so that it is opposite to the one induced by the orientation of W in a neighborhood of u .

By (8.3.ii), $g_{e_1} \dots g_{e_n} = 1$. The n -linear form

$$L_{g_{e_1}} \otimes \dots \otimes L_{g_{e_n}} \rightarrow K$$

defined by $(a_1, \dots, a_n) \mapsto \eta(a_1 \dots a_n, 1_L)$ with $a_r \in L_{g_{e_r}}$ for $r = 1, \dots, n$ is invariant under cyclic permutations and therefore is uniquely determined by u . The tensor product of these forms over all vertices of Γ yields a homomorphism, $D_g : \otimes_{e \in \text{Edg}(\Gamma)} L_{g_e} \rightarrow K$. Set $\tau(W) = \langle g \rangle = D_g(B_g) \in K$. It is clear that for the skeleton dual to a CW-decomposition of W this definition is equivalent to the one given in Section 7.3.

We claim that $\langle g \rangle$ does not depend on the choice of g and Γ . This implies Claim 7.3.1. It suffices to check that $\langle g \rangle$ is preserved under the basic moves and the homotopy moves. It follows directly from definitions and (7.1.d) that $\langle g \rangle$ is invariant under the loop move. The invariance of $\langle g \rangle$ under the contraction move follows from definitions and Lemma 7.1.3. This implies the invariance of $\langle g \rangle$ under the biangle move which is a composition of a loop move and an inverse contraction move.

Assume that two π -systems g_1, g_2 on Γ are related by a homotopy move at a component U of $W \setminus \Gamma$. We claim that it is possible to relate g_1 to g_2 by basic moves. This will imply that $\langle g_1 \rangle = \langle g_2 \rangle$. To prove our claim, take a small subarc f on an edge of Γ adjacent to U . Let A, B be the endpoints of f . Let us connect the points A, B by a small arc, f' , lying inside U (except at its endpoints A, B). The arc f' splits the disc U into a biangle bounded by $f \cup f'$ and a complementary open 2-disc denoted D . It is clear that $\Gamma' = \Gamma \cup f'$ is a skeleton of W obtained from Γ by the biangular move. This move transforms g_1, g_2 into certain π -systems g'_1, g'_2 on Γ' which can be chosen to be related by a homotopy move at the component D of $W \setminus \Gamma'$. Now we transform Γ' as follows. Let us deform f' inside U keeping the endpoint A and gradually pushing the endpoint B along the edges adjacent to U . We do this until B traverses the whole boundary of U and comes back to the original edge from the other side of A . At the end of this deformation we obtain a skeleton Γ'' isotopic to Γ' . Note that while B moves along an edge, both π -systems g'_1, g'_2 move along in the obvious way. When B moves across a vertex of Γ' adjacent to U , the skeleton under deformation is transformed via a contraction move and an inverse contraction move. Under these two moves, the π -systems g'_1, g'_2 are transformed in a canonical way remaining related by a homotopy move at the image of D . At the end of the deformation we obtain π -systems, g''_1, g''_2 on Γ'' related by a homotopy move at the image of D under the deformation. This image is a biangle. Applying the inverse biangular move we transform g''_1, g''_2 into one and the same π -system on Γ . This relates g_1 to g_2 by a sequence of basic moves.

8.5. Construction of a $(1 + 1)$ -dimensional HQFT via skeletons. The construction follows along the same lines as in Section 7.4.

Step 1. We assign K -modules to the so-called split 1-dimensional X -manifolds. A connected non-empty 1-dimensional X -manifold M is *split* if it is

provided with a finite set of points x_1, \dots, x_n with $n \geq 1$ and elements $h_1, \dots, h_n \in \pi$ such that: the points x_1, \dots, x_n are distinct from each other and from the base point; starting from the base point and moving in the direction given by the orientation of M we meet consecutively x_1, \dots, x_n ; the product $h_1 \dots h_n \in \pi$ is the homotopy class of the loop represented by the characteristic map $M \rightarrow X = K(\pi, 1)$. We call x_1, \dots, x_n *split points* and think of each h_r as being attached to x_r . We assign to M the module $A(M) = \otimes_{r=1}^n L_{h_r}$. A non-connected 1-dimensional X -manifold M is *split* if its connected components are split. We define $A(M)$ as the tensor product of the modules A corresponding to the connected components. If $M = \emptyset$ then by definition M is split and $A(M) = K$.

Note that each trivialization (T, g) of a 1-dimensional X -manifold M in the sense of Section 7.4 gives rise to a dual splitting of M . Namely, as split points we take centers of the edges of T . With the center of a (canonically) oriented edge e we associate the element $g_e \in \pi$.

Step 2. Consider a 2-dimensional X -cobordism (W, M_0, M_1) with split bases M_0, M_1 . Let Γ be a skeleton of W and g a π -system on Γ . We say that the pair (Γ, g) *extends the splitting* of ∂W if

- (i) the set of feet of Γ coincides with the set of split points of ∂W ;
- (ii) for each split point $x \in M_0$ (resp. $x \in M_1$) the value of g on the edge of Γ incident to x and directed inside (resp. outside) W is equal to the element of π attached to x .

Choose a pair (Γ, g) extending the splitting of ∂W and such that the map $\tilde{g} : W \rightarrow X$ is homotopic to the characteristic map $W \rightarrow X$. Applying the constructions of Section 8.4 to g we obtain a homomorphism $\langle g \rangle : A(M_0) \rightarrow A(M_1)$. This homomorphism does not depend on the choice of g and Γ . The proof goes exactly as in Section 8.4; the only additional ingredient is the fact that any two homotopic π -systems on Γ extending the same splitting of ∂W can be obtained from each other by homotopy moves at those components of $W \setminus \Gamma$ which are not adjacent to ∂W . This follows from definitions and the fact that each component of ∂W has a base point.

It is clear that the definition of $\langle g \rangle$ is equivalent to the one in Section 7.4 via the passage from trivializations of 1-dimensional X -manifolds to the dual splittings and the passage from CW-decompositions of 2-dimensional X -cobordisms to the dual skeletons. Therefore, the independence of $\langle g \rangle$ on the choice of Γ and g implies the similar claim made in Section 7.4 at Step 2.

Set $\tau(W) = \langle g \rangle : A(M_0) \rightarrow A(M_1)$. For completeness, we state an analogue of Lemma 7.4.1.

8.5.1. Lemma. *Let M_0, M_1, N be split 1-dimensional X -manifolds (possibly void). If a 2-dimensional X -cobordism (W, M_0, M_1) is obtained from two 2-dimensional X -cobordisms (W_0, M_0, N) and (W_1, N, M_1) by gluing along N then $\tau(W) = \tau(W_1) \circ \tau(W_0) : A(M_0) \rightarrow A(M_1)$.*

Proof. For $r = 0, 1$, choose a skeleton Γ_r of W_r and a π -system g_r on Γ_r such

that (Γ_r, g_r) extends the splitting of ∂W_r . Gluing Γ_0 and Γ_1 along their feet lying on N we obtain a skeleton, Γ , of W . Let g be the unique π -system on Γ extending g_0 and g_1 . The lemma follows from the equality $\langle g \rangle = \langle g_1 \rangle \circ \langle g_0 \rangle : A(M_0) \rightarrow A(M_1)$. This equality directly follows from the definition of $\langle g \rangle$.

Step 3. We repeat the constructions made in Section 7.4 at Step 3 replacing everywhere the words “trivialized, trivialization” with “split, splitting”. This gives a $(1+1)$ -dimensional HQFT $(\hat{A}, \hat{\tau})$ with target X .

The duality between the skeletons and the regular CW-decompositions establishes an equivalence between the construction of this section and the one of Section 7. Therefore the HQFT obtained via skeletons coincides with the one constructed in Section 7.

8.6. Theorem. *Let L be a biangular π -algebra over an algebraically closed field K of characteristic 0. Then the associated $(1+1)$ -dimensional HQFT $(\hat{A}, \hat{\tau})$ is semi-cohomological.*

The proof is based on the following lemma.

8.6.1. Lemma. *Let A be a finite-dimensional algebra over a field K and $\eta : A \otimes A \rightarrow K$ be the bilinear form defined by $\eta(a, b) = \text{Tr}(x \mapsto abx : A \rightarrow A)$ where $a, b, x \in A$. If η is non-degenerate then A is semisimple.*

Proof. It suffices to prove that the radical $J(A) \subset A$ of A is zero (see [Hu] for the relevant definitions). If $a \in J(A)$ then for any $b \in A$ there is $n \geq 1$ such that $(ab)^n = 0$. The homomorphism $x \mapsto abx : A \rightarrow A$ is nilpotent and therefore its trace is equal to 0. Hence, $\eta(a, b) = 0$ for any $b \in A$. By assumption, η is non-degenerate, so that $a = 0$.

Proof of Theorem 8.6. We first compute the π -center C of L , cf. Section 7.1. (Here we need no assumptions on K .) Let (A, τ) be the $(1+1)$ -dimensional “HQFT” with target $X = K(\pi, 1)$ constructed at Steps 1 and 2 in Section 8.5. Recall that this “HQFT” is defined for split 1-dimensional X -manifolds and 2-dimensional X -cobordisms with split boundary. By definition, $(\hat{A}, \hat{\tau})$ is the $(1+1)$ -dimensional HQFT with target $X = K(\pi, 1)$ derived from (A, τ) at Step 3 in Section 8.5. For $\alpha \in \pi$, denote by S_α^1 a pointed circle endowed with a map to X representing α . To compute the module

$$C_\alpha = \hat{A}_{S_\alpha^1}$$

we first provide S_α^1 with a splitting, cf. Section 8.5. As a splitting of S_α^1 we take any non-base point $s \in S^1$ endowed with α . By definition,

$$A_{(S_\alpha^1, s)} = L_\alpha.$$

Now, the X -annulus $C_{-+}(\alpha; 1) = S_\alpha^1 \times [0, 1]$ (cf. Section 4.6) is an X -cobordism between two copies of the split circle (S_α^1, s) . The simplest skeleton of $C_{-+}(\alpha; 1)$

extending the splitting of the boundary consists of the arc $s \times [0, 1]$ and an arc connecting two distinct points of $s \times [0, 1]$ and winding once around the annulus. A direct computation from definitions shows that the K -homomorphism $\tau(C_{-+}(\alpha; 1)) : L_\alpha \rightarrow L_\alpha$ is the homomorphism ψ_1 defined in Section 7.1. Hence $C_\alpha = \psi_1(L_\alpha) \subset L_\alpha$. We leave it to the reader to check that multiplication in $C \subset L$ derived from the HQFT $(\hat{A}, \hat{\tau})$ coincides with multiplication in L .

Now we can prove Theorem 8.6. By Lemma 8.6.1, the algebra L_1 is semisimple. Since K is an algebraically closed field, L_1 is a direct sum of matrix rings over K . For a matrix ring $L_1 = \text{Mat}_n(K)$ with $n \geq 1$, the bilinear form (7.1.b) and the corresponding homomorphism $\psi_1 : L_1 \rightarrow L_1$ can be computed explicitly. This computation shows that ψ_1 is a projection of $\text{Mat}_n(K)$ onto its 1-dimensional center. If L_1 is a direct sum of m matrix rings then ψ_1 is a projection of L_1 onto its m -dimensional center K^m . Hence $C_1 = K^m$ and the crossed π -algebra C is semisimple. By Theorem 4.3, the $(1+1)$ -dimensional HQFT $(\hat{A}, \hat{\tau})$ is semi-cohomological.

9. Non-degenerate π -algebras and lattice HQFT's

Throughout this section the group π is *finite* unless explicitly stated to the contrary.

9.1. Non-degenerate π -algebras. Let $L = \oplus_{\alpha \in \pi} L_\alpha$ be a π -algebra. Recall that for $\ell \in L$, we denote by μ_ℓ the left multiplication by ℓ sending any $a \in L$ into $\ell a \in L$. We say that L is *non-degenerate* if the bilinear form $\eta : L \otimes L \rightarrow K$ defined by $\eta(a, b) = \text{Tr}(\mu_{ab} : L \rightarrow L)$ with $a, b \in L$ is non-degenerate. The trace in this formula is well defined since the underlying K -module of L is projective of finite type; here we use the assumption that π is finite. The usual properties of the trace imply that η is symmetric and that (L, η) is a Frobenius π -algebra.

For instance, a biangular π -algebra L is non-degenerate provided (π is finite and) the order $|\pi|$ of π is invertible in K . Note that the inner product $\eta(a, b) = \text{Tr}(\mu_{ab})$ is equal to $|\pi|$ times the inner product considered in Section 7.1.

The direct sums and tensor products of non-degenerate π -algebras are non-degenerate. The push-forward along a homomorphism of finite groups $q : \pi' \rightarrow \pi$ transforms a non-degenerate π' -algebra into a non-degenerate π -algebra. If q is surjective and $|\text{Ker } q|$ is invertible in K , then the pull-back along q of a non-degenerate π -algebra is a non-degenerate π' -algebra.

We shall show that each non-degenerate π -algebra L gives rise to a lattice $(1+1)$ -dimensional HQFT with target $K(\pi, 1)$. The underlying crossed π -algebra, $C = \oplus_{\alpha \in \pi} C_\alpha$, of this HQFT is called the π -center of L . (In the case where L is biangular, this definition is equivalent to the one in Section 7.1 up to rescaling, cf. Remark 9.7.1.) We give here an algebraic description of C .

Observe first that for every $\alpha \in \pi$, the form $\eta : L_\alpha \otimes L_{\alpha^{-1}} \rightarrow K$ defined above is non-degenerate and yields a canonical element $b_\alpha \in L_\alpha \otimes L_{\alpha^{-1}}$ (cf. Lemma 7.1.1). As in Section 7.1, we have an expansion $b_\alpha = \sum_i p_i^\alpha \otimes q_i^\alpha$. The expression

$\hat{b}_\alpha = \sum_i p_i^\alpha q_i^\alpha \in L_1$ does not depend on the choice of this expansion. We claim that

$$(9.1.a) \quad \sum_{\alpha \in \pi} \hat{b}_\alpha = 1_L.$$

To prove it, observe that for any $\ell \in L_1$,

$$\begin{aligned} \eta(\ell, 1_L) &= \text{Tr}(\mu_\ell : L \rightarrow L) = \sum_{\alpha \in \pi} \text{Tr}(\mu_\ell|_{L_\alpha} : L_\alpha \rightarrow L_\alpha) \\ &= \sum_{\alpha \in \pi} \sum_i \eta(\ell p_i^\alpha, q_i^\alpha) = \sum_{\alpha \in \pi} \eta(\ell, \sum_i p_i^\alpha q_i^\alpha) = \sum_{\alpha \in \pi} \eta(\ell, \hat{b}_\alpha) = \eta(\ell, \sum_{\alpha \in \pi} \hat{b}_\alpha) \end{aligned}$$

where we use the expression for the trace given in Lemma 7.1.1. Since the form $\eta : L_1 \otimes L_1 \rightarrow K$ is non-degenerate, $\sum_{\alpha \in \pi} \hat{b}_\alpha = 1_L$.

As in Section 7.1, we use $b_\alpha = \sum_i p_i^\alpha \otimes q_i^\alpha$ to define a K -homomorphism $\psi_\alpha : L \rightarrow L$ by $\psi_\alpha(a) = \sum_i p_i^\alpha a q_i^\alpha$ where $a \in L$. This homomorphism does not depend on the choice of the expansion of b_α and sends each $L_\beta \subset L$ into $L_{\alpha\beta\alpha^{-1}}$.

For $\omega \in \pi$, we can push (L, η) back along the conjugation $\alpha \mapsto \omega\alpha\omega^{-1} : \pi \rightarrow \pi$. This gives a Frobenius π -algebra L^ω such that $L_\alpha^\omega = L_{\omega\alpha\omega^{-1}}$ for all $\alpha \in \pi$. Let P be the direct sum of the Frobenius π -algebras L^ω over all $\omega \in \pi$. Thus,

$$P_\alpha = \bigoplus_{\omega \in \pi} L_{\omega\alpha\omega^{-1}}$$

for all $\alpha \in \pi$. Given $\beta \in \pi$, consider a homomorphism $\Psi_\beta : P_\alpha \rightarrow P_{\beta\alpha\beta^{-1}}$ defined by the block-matrix $[\psi_{\omega'\beta\omega^{-1}}]_{\omega, \omega' \in \pi}$. Thus, for $a \in L_{\omega\alpha\omega^{-1}} \subset P_\alpha$ we have

$$\Psi_\beta(a) = \bigoplus_{\omega' \in \pi} \psi_{\omega'\beta\omega^{-1}}(a) \in \bigoplus_{\omega' \in \pi} L_{\omega'\beta\alpha\beta^{-1}(\omega')^{-1}} = P_{\beta\alpha\beta^{-1}}.$$

It turns out that $\Psi_\beta \Psi_{\beta'} = \Psi_{\beta\beta'}$ for any $\beta, \beta' \in \pi$ and Ψ_β is adjoint to $\Psi_{\beta^{-1}}$ with respect to the inner product in P . In particular, Ψ_1 is an orthogonal projection onto a submodule $C = \bigoplus_{\alpha \in \pi} C_\alpha$ of P where $C_\alpha = \Psi_1(P_\alpha) \subset P_\alpha$. Formula (9.1.a) implies that $1_P \in C$. It turns out that C is a subalgebra of P . Moreover, the inner product in P restricted to C and the homomorphism $\pi \rightarrow \text{Aut} C$ sending each $\beta \in \pi$ to $\Psi_\beta|_C : C \rightarrow C$ make C a crossed π -algebra. The crossed π -algebra C is the π -center of L . All these claims follow directly by applying the definitions of Section 4 to the HQFT associated with L (cf. Section 9.6).

9.1.1. Remark. It is curious to note (we shall not use it) that a non-degenerate π -algebra L is separable. Recall that a unital associative K -algebra A is separable if there is $b \in A \otimes A$ such that (i) the algebra multiplication $A \otimes A \rightarrow A$ sends b into 1 and (ii) for any $a \in A$, $(a \otimes 1)b = b(1 \otimes a)$ (see [Pi, Chapter 10]). Such an element b is called a separating idempotent of A . We claim that $b = \sum_{\alpha \in \pi} b_\alpha \in L \otimes L$ is

a separating idempotent of L . Condition (i) follows from (9.1.a), Condition (ii) follows from the fact that for any $\alpha, \beta \in \pi, a \in L_\alpha$,

$$(9.1.b) \quad (a \otimes 1) b_\beta = b_{\alpha\beta} (1 \otimes a).$$

Summing over all β , we obtain $(a \otimes 1)b = b(1 \otimes a)$. To prove (9.1.b), it suffices to show that for any $x \in L_{\beta^{-1}\alpha^{-1}}$,

$$\sum_i \eta(ap_i^\beta, x) q_i^\beta = \sum_j \eta(p_j^{\alpha\beta}, x) q_j^{\alpha\beta} a.$$

Using $\eta(ap_i^\beta, x) = \eta(xa, p_i^\beta)$, we easily observe that both sides are equal to xa .

9.2. π -systems on CW-complexes re-examined. Let T be a regular finite CW-complex with underlying topological space \tilde{T} and a distinguished set of base vertices. Let F_T be the group formed by the maps $\text{Vert}(T) \rightarrow \pi$ taking value $1 \in \pi$ on all base vertices of T . Clearly, $F_T = \pi^{n_T}$ where n_T is the number of vertices of T distinct from the base vertices. Each homotopy class G of π -systems on T is an orbit of the action of F_T on the set of π -systems on T , see Section 7.2. This implies that $\text{card}(G)$ is an integer divisor of $|F_T| = |\pi|^{n_T}$. More precisely, $|\pi|^{n_T} / \text{card}(G) = |\text{Stab}_g|$ where $\text{Stab}_g \subset F_T$ is the stabilizer of an element $g \in G$. If \tilde{T} is connected and the set of base vertices is non-void then $\text{Stab}_g = 1$ so that $\text{card}(G) = |\pi|^{n_T}$. If \tilde{T} is connected and the set of base vertices is empty then the group Stab_g can be computed in homotopy terms. Consider the map $\tilde{g} : \tilde{T} \rightarrow X = K(\pi, 1)$ determined by $g \in G$ and the induced homomorphism $\tilde{g}_\# : \pi_1(\tilde{T}) \rightarrow \pi$. Then Stab_g is isomorphic to the subgroup of π consisting of the elements of π commuting with all elements of $\tilde{g}_\#(\pi_1(\tilde{T})) \subset \pi$.

In Section 9.4 we shall need a notion of enriched π -systems on T . By an *enriched π -system* on T we mean a pair (ω, g) where ω is an arbitrary map from the set of base vertices of T to π and g is a π -system on T . Given an enriched π -system (ω, π) , we define a π -system ωg on T as follows: extend ω to all vertices of T by assigning $1 \in \pi$ to all non-base vertices and set $(\omega g)_e = \omega(i_e) g_e (\omega(t_e))^{-1}$ for any $e \in \text{Edg}(T)$. The induced map $\tilde{\omega}g : \tilde{T} \rightarrow X$ is obtained from $\tilde{g} : \tilde{T} \rightarrow X$ by pushing the image of each base vertex $z \in T$ along a loop representing $\omega(z) \in \pi$. The homomorphisms $\pi_1(T, z) \rightarrow \pi$ induced by \tilde{g} and $\tilde{\omega}g$ are conjugated by $\omega(z) \in \pi$. Hence the π -systems g and ωg are not homotopic unless $\omega = 1$.

If T is a CW-decomposition of an X -manifold or an X -cobordism then an enriched π -system (ω, g) on T is *characteristic* if the π -system g is characteristic.

9.3. State sums on closed X -surfaces. Fix a non-degenerate π -algebra L . Let W be a closed oriented surface endowed with a map $W \rightarrow X = K(\pi, 1)$. Here we define a state sum invariant $\tau(W) = \tau_L(W) \in K$. Choose a regular CW-decomposition T of W . Let G be the homotopy class of characteristic π -systems on T . Repeating the definitions of Section 7.3 for $g \in G$ we obtain a state sum $\langle g \rangle \in K$. In general this state sum depends on the choice of g in G .

9.3.1. Claim. *The sum $\sum_{g \in G} \langle g \rangle$ does not depend on the choice of T .*

We outline a proof in Section 9.5. Set $\langle W \rangle_L = \sum_{g \in G} \langle g \rangle \in K$. Claim 9.3.1 implies that $\langle W \rangle_L$ is a well defined invariant of the closed X -surface W . To include this state sum invariant into an HQFT, we need to renormalize it (cf. the proof of Lemma 9.4.1). Set

$$(9.3.a) \quad \tau(W) = \frac{|\pi|^{n_T}}{\text{card}(G)} \langle W \rangle_L = \frac{|\pi|^{n_T}}{\text{card}(G)} \sum_{g \in G} \langle g \rangle \in K$$

where n_T is the number of vertices of T . According to the results of Section 9.2, the number $|\pi|^{n_T}/\text{card}(G)$ is an integer. This number is multiplicative with respect to disjoint union of closed X -surfaces and can be computed in homotopy terms, see Section 9.2. In particular, this number is independent of T and L . Therefore $\tau(W)$ is a well defined invariant of W . By the very definition, $\tau(W)$ depends only on the homotopy class of the characteristic map $W \rightarrow X$ and is multiplicative with respect to disjoint union of closed X -surfaces.

It is useful to rewrite the definition of $\tau(W)$ in terms of the group $F_T = \pi^{n_T}$ formed by the maps $\text{Vert}(T) \rightarrow \pi$ and its action on the set of π -structures on T . Namely, for any characteristic π -system $g \in G$ on T , we have

$$\tau(W) = \sum_{\gamma \in F_T} \langle \gamma g \rangle.$$

We can give an explicit formula for $\tau(W)$ when W is a closed connected oriented surface of genus $n \geq 1$. Using the same CW-decomposition of W and the same notation as in Section 7.3, we obtain

$$\tau(W) = \sum_{\beta \in \pi} \eta \left(\prod_{r=1}^n [b_{\beta \alpha_{2r-1} \beta^{-1}}, b_{\beta \alpha_{2r} \beta^{-1}}], 1_L \right)$$

where $\prod_{r=1}^n [b_{\beta \alpha_{2r-1} \beta^{-1}}, b_{\beta \alpha_{2r} \beta^{-1}}] \in L_1$. Similarly,

$$\tau(S^2) = |\pi| \eta(1_L, 1_L) = |\pi| \text{Dim} L.$$

9.4. Lattice construction of a $(1+1)$ -dimensional HQFT. We extend the invariant of closed X -surfaces constructed in Section 9.3 to a $(1+1)$ -dimensional HQFT with target $X = K(\pi, 1)$. The construction follows the same lines as in Section 7.4.

Step 1. Let M be a 1-dimensional X -manifold with distinguished CW-decomposition T (such that the base points of the components of M are among the vertices). We provide each edge e of M with canonical orientation induced by

the one of M . For a π -system g on T , we define a module $A(M, g) = \otimes_e L_{g_e}$ where e runs over all canonically oriented edges of T . Set

$$(9.4.a) \quad A(M) = \bigoplus_{(\omega, g)} A(M, \omega g)$$

where (ω, g) runs over all characteristic enriched π -systems on T (so that $\tilde{g} : M \rightarrow X$ is homotopic to the characteristic map $M \rightarrow X$). Since the number of such systems is finite, the module $A(M)$ is projective of finite type. It is clear that for disjoint 1-dimensional X -manifolds M, N endowed with CW-decompositions, we have $A(M \amalg N) = A(M) \otimes A(N)$. If $M = \emptyset$ then by definition M has a unique enriched π -system and $A(M) = K$.

The module $A(M, \omega g)$ can be explicitly computed as follows. Assume first that $M = S^1$. Starting from the base point, $z \in M$, and moving along M in the direction determined by the orientation of M we meet consecutively the oriented edges, e_1, \dots, e_n , of T where $n \geq 1$ is the number of edges of T . Then

$$A(M, \omega g) = L_{\omega(z)g_{e_1}} \otimes (\otimes_{r=2}^{n-1} L_{g_{e_r}}) \otimes L_{g_{e_n}(\omega(z))^{-1}}.$$

If M has several components then restricting (ω, g) we obtain enriched π -systems on these components and $A(M, \omega g)$ is the tensor product of the corresponding modules.

Step 2. Consider a 2-dimensional X -cobordism (W, M_0, M_1) . Assume that at least one of the bases M_0, M_1 is non-void and fix a CW-decomposition of the bases. Using (9.4.a), we define a K -homomorphism $\tau(W) : A(M_0) \rightarrow A(M_1)$ by a block-matrix of homomorphisms $\tau(W; (\omega_0, g_0), (\omega_1, g_1)) : A(M_0, \omega_0 g_0) \rightarrow A(M_1, \omega_1 g_1)$ where (ω_r, g_r) runs over characteristic enriched π -systems on the given CW-decomposition of M_r , for $r = 0, 1$. Fix a pair $(\omega_0, g_0), (\omega_1, g_1)$. Choose a CW-decomposition T of W extending the given CW-decomposition of $\partial W = (-M_0) \cup M_1$. Denote by G the set of characteristic π -systems g on T extending $g_0 \cup g_1$. The set G is non-void and finite. It is clear that for any $g \in G$ the pair $(\omega = \omega_0 \cup \omega_1, g)$ is an enriched π -system on T . Consider the π -system ωg on T extending the π -system $(\omega_0 g_0) \cup (\omega_1 g_1)$ on the boundary. Repeating the definitions of Section 7.4 for ωg we obtain a homomorphism $\langle \omega g \rangle : A(M_0, \omega_0 g_0) \rightarrow A(M_1, \omega_1 g_1)$. Set

$$\tau(W; (\omega_0, g_0), (\omega_1, g_1)) = \sum_{g \in G} \langle \omega g \rangle : A(M_0, \omega_0 g_0) \rightarrow A(M_1, \omega_1 g_1).$$

We claim that this homomorphism is independent of the choice of T ; for a proof, see Section 9.5.

By the very definition, $\tau(W)$ depends only on the homotopy class of the characteristic map $W \rightarrow X$ and satisfies (1.2.5).

9.4.1. Lemma. *Let M_0, M_1, N be 1-dimensional X -manifolds (possibly void) with fixed CW-decompositions. If a 2-dimensional X -cobordism (W, M_0, M_1) is*

obtained from two 2-dimensional X -cobordisms (W_0, M_0, N) and (W_1, N, M_1) by gluing along N then $\tau(W) = \tau(W_1) \circ \tau(W_0) : A(M_0) \rightarrow A(M_1)$.

Proof. Because of the multiplicativity of τ with respect to disjoint union, it suffices to consider the case where W is connected and $N \neq \emptyset$. For $r = 0, 1$, fix a regular CW-decomposition T_r of W_r extending the given CW-decomposition of the boundary. Gluing T_0 and T_1 along N we obtain a regular CW-decomposition, T , of W .

Assume first that at least one of the manifolds M_0, M_1 is non-void. Fix a characteristic enriched π -system (ω_r, g_r) on the given CW-decomposition of M_r for $r = 0, 1$. We should prove that

$$(9.4.b) \quad \tau(W; (\omega_0, g_0), (\omega_1, g_1)) = \sum_y \tau(W_1; y, (\omega_1, g_1)) \circ \tau(W_0; (\omega_0, g_0), y)$$

where y runs over all characteristic enriched π -systems on N .

Fix a characteristic π -system h on N . Choose a characteristic π -system H_r on T_r extending $g_r \cup h$. It is clear that there is a unique π -system, H , on T such that $H|_{T_r} = H_r$ for $r = 0, 1$. The corresponding map $\tilde{H} : W \rightarrow X$ is homotopic to the map obtained by gluing the characteristic maps $W_0 \rightarrow X, W_1 \rightarrow X$ along N .

Denote by F be the group formed by the maps $\text{Vert}(T) \rightarrow \pi$ taking the value $1 \in \pi$ on all vertices of T lying in ∂W . We introduce four subgroups F_0, F_1, F_N, F_b of F as follows. For $r = 0, 1$, the group F_r consists of $\gamma \in F$ such that $\gamma(u) = 1 \in \pi$ for all vertices u of T except possibly those lying in $W_r \setminus \partial W_r$. The group F_N consists of $\gamma \in F$ such that $\gamma(u) = 1 \in \pi$ for all vertices u of T except possibly those lying in N and distinct from the base vertices of N . The group F_b consists of $\gamma \in F$ such that $\gamma(u) = 1 \in \pi$ for any vertex u of T distinct from the base vertices of N .

The subgroups F_0, F_1, F_N, F_b of F commute with each other. Every $\gamma \in F$ splits uniquely as the product $\gamma = \gamma_0 \gamma_1 \gamma_N \gamma_b$ with $\gamma_0 \in F_0, \gamma_1 \in F_1, \gamma_N \in F_N, \gamma_b \in F_b$.

By definition,

$$\tau(W; (\omega_0, g_0), (\omega_1, g_1)) = \sum_{g \in G} \langle \omega g \rangle$$

where $\omega = \omega_0 \cup \omega_1$ and G is the set of characteristic π -systems on T extending $g_0 \cup g_1$. Thus, every $g \in G$ is homotopic to H and coincides with H on ∂W . The definition of homotopy for π -systems (see Section 7.2) and the assumption that each component of ∂W contains a base point imply that G is the orbit of H under the action of F on the set of π -systems on T . Since $\partial W \neq \emptyset$, this action is fixed point free and therefore

$$\begin{aligned} \tau(W; (\omega_0, g_0), (\omega_1, g_1)) &= \sum_{g \in G} \langle \omega g \rangle = \sum_{\gamma \in F} \langle \omega \gamma H \rangle \\ &= \sum_{\gamma_0 \in F_0} \sum_{\gamma_1 \in F_1} \sum_{\gamma_N \in F_N} \sum_{\gamma_b \in F_b} \langle \omega \gamma_0 \gamma_1 \gamma_N \gamma_b H \rangle : A(M_0, \omega_0 g_0) \rightarrow A(M_1, \omega_1 g_1). \end{aligned}$$

The claim (*) in the proof of Lemma 7.4.1 directly implies that the homomorphism $\langle \omega \gamma_0 \gamma_1 \gamma_N \gamma_b H \rangle$ is the composition of the homomorphisms

$$\langle \omega_0 \gamma_0 \gamma_N \gamma_b H_0 \rangle : A(M_0, \omega_0 g_0) \rightarrow A(N, \gamma_N \gamma_b h)$$

and

$$\langle \omega_1 \gamma_1 \gamma_N \gamma_b H_1 \rangle : A(N, \gamma_N \gamma_b h) \rightarrow A(M_1, \omega_1 g_1).$$

Summing over $\gamma_0 \in F_0, \gamma_1 \in F_1$ we obtain that

$$\begin{aligned} & \tau(W; (\omega_0, g_0), (\omega_1, g_1)) = \\ &= \sum_{\gamma_N \in F_N} \sum_{\gamma_b \in F_b} \tau(W_1; (\gamma_b, \gamma_N h), (\omega_1, g_1)) \circ \tau(W_0; (\omega_0, g_0), (\gamma_b, \gamma_N h)). \end{aligned}$$

Finally note that when γ_N runs over F_N and γ_b runs over F_b the enriched π -system $y = (\gamma_b, \gamma_N h)$ runs over all characteristic enriched π -systems on N . This gives (9.4.b).

In the case $M_0 = M_1 = \emptyset$ the same argument gives

$$\tau(W) = \sum_{\gamma \in F} \langle \gamma H \rangle = \tau(W_1) \circ \tau(W_0).$$

Step 3. We apply the constructions given at Step 3 in Section 7.4 to the preliminary HQFT defined above. It suffices to replace everywhere the word “trivialized” with “provided with a CW-decomposition”. This gives a $(1+1)$ -dimensional HQFT $(\hat{A}, \hat{\tau})$ associated with the non-degenerate π -algebra L . Note that for any closed X -surface W we have $\hat{\tau}(W) = \tau(W)$.

9.5. Proof of Claim 9.3.1. The proof goes along the same lines as the proof of Claim 7.3.1 given in Section 8. One translates the state sum in terms of the skeletons and proves the invariance under the local moves discussed in Section 8.1. The invariance under the contraction move directly follows from definitions and Lemma 7.1.3 which applies to L without any changes. Consider the loop move $\Gamma \mapsto \Gamma'$. Every π -system g on Γ lifts to a finite family, $Y(g)$, of π -systems on Γ' . (In fact, $\text{card}(Y(g)) = |\pi|$). It follows from definitions and (9.1.a) that $\langle g \rangle = \sum_{g' \in Y(g)} \langle g' \rangle$. This implies the invariance of the sum $\sum_{g \in G} \langle g \rangle$ under the loop move. In contrast to the constructions of Section 7, we do not need to prove the invariance under homotopy moves.

The case of X -cobordisms is similar.

9.6. Theorem. *Let L be a non-degenerate π -algebra over an algebraically closed field K of characteristic 0. Then the associated $(1+1)$ -dimensional HQFT $(\hat{A}, \hat{\tau})$ is semi-cohomological.*

Proof. Let $\alpha \in \pi$ and $b_\alpha = \sum_i p_i^\alpha \otimes q_i^\alpha \in L_\alpha \otimes L_{\alpha^{-1}}$ be the canonical element associated with the restriction of the inner product η (defined in Section 9.1) onto $L_\alpha \otimes L_{\alpha^{-1}}$. Recall the homomorphism $\psi_\alpha : L \rightarrow L$ defined by $\psi_\alpha(a) = \sum_i p_i^\alpha a q_i^\alpha$ where $a \in L$. Observe that for any $b \in L_1$ and $x \in L_{\alpha^{-1}}$,

$$\sum_i \eta(bp_i^\alpha, x) q_i^\alpha = \sum_i \eta(xb, p_i^\alpha) q_i^\alpha = xb = \sum_i \eta(x, p_i^\alpha) q_i^\alpha b = \sum_i \eta(p_i^\alpha, x) q_i^\alpha b.$$

Therefore

$$\sum_i bp_i^\alpha \otimes q_i^\alpha = \sum_i p_i^\alpha \otimes q_i^\alpha b.$$

This implies that for any $a, b \in L_1$,

$$b\psi_\alpha(a) = \sum_i bp_i^\alpha a q_i^\alpha = \sum_i p_i^\alpha a q_i^\alpha b = \psi_\alpha(a)b.$$

In other words, $\psi_\alpha(L_1)$ lies in the center, $Z(L_1)$, of L_1 for all $\alpha \in \pi$.

Recall the module P_α introduced in Section 9.1. Each $a \in P_\alpha$ splits uniquely as a sum $a = \sum_{\omega \in \pi} a_\omega$ with $a_\omega \in L_{\omega\alpha\omega^{-1}}$. We view $\{a_\omega\}_\omega$ as the coordinates of a . Computations similar to the ones in the proof of Theorem 8.6 give the algebraic description of the π -center C of L formulated in Section 9.1. In particular, $C_\alpha \subset P_\alpha$ is the image of the projection $\Psi_1 : P_\alpha \rightarrow P_\alpha$ given in the coordinates by the block-matrix $[\psi_{\omega'\omega^{-1}}]_{\omega, \omega' \in \pi}$. An element $a \in P_\alpha$ lies in C_α if and only if for all $\omega' \in \pi$,

$$a_{\omega'} = \sum_{\omega \in \pi} \psi_{\omega'\omega^{-1}}(a_\omega).$$

Taking $\alpha = 1$ and applying the result of the preceding paragraph we obtain that

$$C_1 \subset Z(L_1)^{|\pi|} \subset P_1 = L_1^{|\pi|}.$$

Using an appropriate skeleton of a disc with two holes, we can compute multiplication in C as follows. If $a \in C_\alpha, b \in C_\beta$, then the ν -th coordinate of $ab \in C_{\alpha\beta}$ corresponding to $\nu \in \pi$ is computed by

$$(ab)_\nu = \sum_{\omega, \mu \in \pi} \psi_{\nu\omega^{-1}}(a_\omega) \psi_{\nu\mu^{-1}}(b_\mu) = \left(\sum_{\omega \in \pi} \psi_{\nu\omega^{-1}}(a_\omega) \right) \left(\sum_{\mu \in \pi} \psi_{\nu\mu^{-1}}(b_\mu) \right) = a_\nu b_\nu.$$

In particular, C_1 is a subalgebra of $P_1 = L_1^{|\pi|}$. By the results above, C_1 is a subalgebra of $Z(L_1)^{|\pi|}$.

The argument given in the proof of Lemma 8.6.1 shows that the algebra L_1 is semisimple. Since K is an algebraically closed field, L_1 is a direct sum of matrix rings over K and its center $Z(L_1)$ is a direct sum of several copies of K . Hence the algebra C_1 is a direct sum of several copies of K and the crossed π -algebra C is semisimple. By Theorem 4.3, the HQFT $(\hat{A}, \hat{\tau})$ is semi-cohomological.

9.7. Remarks. 1. Let L be a biangular π -algebra. If $|\pi|$ is invertible in K then L is non-degenerate and we have two $(1+1)$ -dimensional HQFT's associated to L : the HQFT (A_1, τ_1) defined in Section 7 and the HQFT (A_2, τ_2) defined in this section. These HQFT's are equivalent up to rescaling. In particular, for a closed X -surface W we have $\tau_2(W) = |\pi|^{\chi(W)} \tau_1(W)$. The factor $|\pi|^{\chi(W)}$ comes from the normalization factor in (9.3.a) and from the fact that the inner products considered in Sections 7.1 and 9.1 differ by a factor of $|\pi|$. It can be shown that $A_1(M) = A_2(M)$ for any 1-dimensional X -manifold M . Thus the construction of an HQFT given in this section generalizes the construction of Section 7 in the case of finite π with $|\pi|$ invertible in K .

2. The HQFT constructed in this section is additive (resp. multiplicative) with respect to direct sums (resp. tensor products) of non-degenerate π -algebras. The invariant τ of closed X -surfaces defined in Section 9.3 is compatible with pull-backs and push-forwards of non-degenerate group-algebras as follows. Consider a group homomorphism $q : \pi' \rightarrow \pi$. Let $f_q : X' = K(\pi', 1) \rightarrow K(\pi, 1) = X$ be the map induced by q . Let $L = q_*(L')$ be the non-degenerate π -algebra obtained as the push-forward along q of a non-degenerate π' -algebra L' . Consider a closed X -surface $(W, g : W \rightarrow X)$. It follows from definitions that $\langle W, g \rangle_L = \sum_{g'} \langle W, g' \rangle_{L'}$ where g' runs over the homotopy classes of maps $W \rightarrow X'$ such that $f_q g'$ is homotopic to g . For connected W , this can be rewritten as follows:

$$\tau_L(W, g) = \sum_{g'} \frac{|\text{Stab}_g|}{|\text{Stab}_{g'}|} \tau_{L'}(W, g')$$

where $|\text{Stab}_g|$ is the number of elements of π commuting with all elements of the image of the homomorphism $\pi_1(W) \rightarrow \pi$ induced by g .

To consider the pull-backs, assume that $q(\pi') = \pi$ and $|\text{Ker } q|$ is invertible in K . Let $q^*(L)$ be the non-degenerate π' -algebra obtained as the pull-back along q of a non-degenerate π -algebra L . For each closed X' -surface $W = (W, g' : W \rightarrow X')$, we have a closed X -surface $W_q = (W, f_q g' : W \rightarrow X)$. It follows from definitions that $\tau_{q^*(L)}(W) = |\text{Ker } q|^{\chi(W)} \tau_L(W_q)$. For instance, consider the trivial homomorphism $q : \pi \rightarrow \{1\}$. If L is the 1-dimensional $\{1\}$ -algebra K , then $q^*(L)$ is the group ring $K[\pi]$ and $\tau_{K[\pi]}(W) = |\pi|^{\chi(W)} \tau_L(W_q) = |\pi|^{\chi(W)}$.

3. It would be interesting to generalize the constructions of this section to π -algebras over an infinite group π . A part of these constructions can be carried over to so-called finite π -algebras. A π -algebra L is *finite* if $L_\alpha = 0$ for all but finitely many $\alpha \in \pi$. The definition of non-degeneracy applies to finite π -algebras word for word. If L is a non-degenerate finite π -algebra over a possibly infinite group π then the state sum $\sum_{g \in G} \langle g \rangle \in K$ in Section 9.3 is finite and gives a well defined invariant of the closed X -surface W . However, the normalization factor $|\pi|^{n_T} / \text{card}(G)$ appearing in Section 9.3 may be infinite and the modules appearing in Section 9.4 may be of infinite type. Another possible approach is to consider topological groups and to replace state sums with integrals.

10. Further examples of group-algebras

In this section we have collected a few miscellaneous examples and constructions of group-algebras.

10.1. Biangular group-algebras from algebra automorphisms. Let A be an associative unital algebra over K whose underlying K -module is projective of finite type. Let π be a group acting on A by algebra automorphisms. Consider the direct sum $L = \bigoplus_{\alpha \in \pi} A\alpha$ of $\text{card}(\pi)$ copies of A numerated by the elements of π . We provide L with multiplication by $(a\alpha)(b\beta) = (a\alpha(b))(\alpha\beta)$ where $a, b \in A$ and $\alpha, \beta \in \pi$. It is easy to check that L is an associative algebra. We provide L with the structure of a π -algebra by $L_\alpha = A\alpha$ for all $\alpha \in \pi$. This π -algebra always satisfies condition (i) in the definition of biangular π -algebras. It satisfies condition (ii) if and only if the bilinear form $\eta : A \otimes A \rightarrow K$ defined by $\eta(a, b) = \text{Tr}(\mu_{ab} : A \rightarrow A)$ is non-degenerate. The form η is non-degenerate for instance when A is a direct sum of matrix rings $\text{Mat}_n(K)$ such that n is invertible in K . This can be easily deduced from the following simple lemma.

10.1.1. Lemma. *Let P, Q be free K -modules of finite rank. Then: (i) the pairing $\text{Hom}(P, Q) \otimes \text{Hom}(Q, P) \rightarrow K$ sending $(f \in \text{Hom}(P, Q), g \in \text{Hom}(Q, P))$ to $\text{Tr}(fg) = \text{Tr}(gf)$ is non-degenerate; (ii) for any $\ell \in \text{Hom}(Q, Q)$, the trace of the endomorphism of $\text{Hom}(P, Q)$ sending any $f \in \text{Hom}(P, Q)$ into $\ell f \in \text{Hom}(P, Q)$ is equal to $\text{Tr}(\ell) \text{Dim}(P)$.*

10.2. More non-degenerate and biangular group-algebras. Let $\{V_s\}_{s \in S}$ be a family of free K -modules of finite rank numerated by elements of a finite set S . With every left action of π on S we associate a π -algebra L as follows. For $\alpha \in \pi$, set

$$L_\alpha = \bigoplus_{s \in S} \text{Hom}(V_s, V_{\alpha(s)}).$$

Each element $a \in L_\alpha$ is determined by its “coordinates” $\{a_s \in \text{Hom}(V_s, V_{\alpha(s)})\}_{s \in S}$. The K -linear structure in L is coordinate-wise. If $a \in L_\alpha, b \in L_\beta$ then the product $ab \in L_{\alpha\beta}$ is defined in coordinates by $(ab)_s = a_{\beta(s)}b_s \in \text{Hom}(V_s, V_{\alpha\beta(s)})$. It is obvious that this multiplication is associative and makes $L = \bigoplus_{\alpha \in \pi} L_\alpha$ a π -algebra. The unit 1_L is determined by $(1_L)_s = \text{id}_{V_s}$ for all $s \in S$. It follows from Lemma 10.1.1 that L is non-degenerate if (and only if) π is finite and either $V_s = 0$ for all $s \in S$ or $\text{Dim}V_s \in K$ is invertible in K for all $s \in S$. It follows from Lemma 10.1.1 that L is biangular if (and only if) either $V_s = 0$ for all $s \in S$ or $\text{Dim}V_s \in K$ does not depend on s and is invertible in K . This gives examples of non-degenerate π -algebras which are not biangular.

If L is biangular then an explicit computation of the homomorphism $\psi_1 : L_1 \rightarrow L_1$ defined in Section 7.1 shows that it is a projection of the algebra $L_1 = \bigoplus_s \text{End}(V_s)$ onto its center $K^{\text{card}(S)}$. Note also that if the action of π on S is free and transitive then $L = \text{End}(\bigoplus_{s \in S} V_s)$.

Replacing the direct sum with tensor product we obtain another interesting

π -algebra. Namely, for $\alpha \in \pi$, set

$$R_\alpha = \bigotimes_{s \in S} \text{Hom}(V_s, V_{\alpha(s)}).$$

The K -module R_α is additively generated by the elements $a = \otimes_s a_s$ where $a_s \in \text{Hom}(V_s, V_{\alpha(s)})$ for $s \in S$. If $b = \otimes_s b_s \in R_\beta$ with $b_s \in \text{Hom}(V_s, V_{\beta(s)})$ then the product $ab \in R_{\alpha\beta}$ is defined by $ab = \otimes_s (ab)_s$ where $(ab)_s = a_{\beta(s)} b_s \in \text{Hom}(V_s, V_{\alpha\beta(s)})$ for $s \in S$. This multiplication is associative and makes $R = \bigoplus_{\alpha \in \pi} R_\alpha$ a π -algebra with unit $\otimes_s \text{id}_{V_s}$. It follows from Lemma 10.1.1 that this π -algebra always satisfies (7.1.i) and satisfies (7.1.ii) if and only if is either $V_s = 0$ for all $s \in S$ or $\text{Dim} V_s \in K$ is invertible in K for all $s \in S$.

Assume that K is a field of characteristic 0 and $V_s \neq 0$ for all $s \in S$. Then R is biangular and the homomorphism $\psi_1 : R_1 \rightarrow R_1$ defined in Section 7.1 is a projection of the algebra $R_1 = \bigotimes_{s \in S} \text{End}(V_s)$ onto its 1-dimensional center. Therefore the lattice HQFT determined by R is obtained by k^{ρ_0} -rescaling from the primitive cohomological HQFT determined by an element of $H^2(\pi; K^*)$ where $k = \eta(1_R, 1_R) = \text{Dim} R_1 = (\prod_s \text{Dim} V_s)^2$.

10.3. Crossed group-algebras via push-forward. Consider a group epimorphism $q : \pi' \rightarrow \pi$ with finite kernel, H , lying in the center of π' . We can push forward any crossed π' -algebra L' along q to obtain a Frobenius π -algebra $L = q_*(L')$, cf. Section 3.1. Recall that $L = L'$ as Frobenius algebras. If $H = \text{Ker } q$ is contained in the kernel of the action φ of π' on L' , then φ induces an action of π on L . We claim that L is a crossed π -algebra. Axioms (3.1.1) - (3.2.3) for L directly follows from the corresponding axioms for L' . Let us check (3.2.4) for L . Let $\alpha, \beta \in \pi$ and $c \in L_{\alpha\beta\alpha^{-1}\beta^{-1}}$. Note that for $u \in q^{-1}(\alpha), v \in q^{-1}(\beta)$, the commutator $uvu^{-1}v^{-1}$ does not depend on the choice of u and v . Denote this distinguished element of $q^{-1}(\alpha\beta\alpha^{-1}\beta^{-1})$ by $w_{\alpha,\beta}$. To check equality (3.2.a), it suffices to consider the case where $c \in L'_w$ where $w \in q^{-1}(\alpha\beta\alpha^{-1}\beta^{-1})$. The homomorphism $c\varphi_\beta = c\varphi_v$ sends each direct summand L'_u of L_α (with $u \in q^{-1}(\alpha)$) into $L'_{wvuv^{-1}}$. Therefore

$$\begin{aligned} \text{Tr}(c\varphi_\beta : L_\alpha \rightarrow L_\alpha) &= \sum_{u \in q^{-1}(\alpha), wvuv^{-1}=u} \text{Tr}(c\varphi_v : L'_u \rightarrow L'_u) \\ &= \begin{cases} \sum_{u \in q^{-1}(\alpha)} \text{Tr}(c\varphi_v : L'_u \rightarrow L'_u), & \text{if } w = w_{\alpha,\beta}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

The assumption $\varphi(H) = 1$ implies that the trace $\text{Tr}(c\varphi_v : L'_u \rightarrow L'_u)$ does not depend on the choice of v in $q^{-1}(\beta)$. The same argument together with formula (3.2.a) shows that this trace does not depend on the choice of u in $q^{-1}(\alpha)$. Hence,

$$\text{Tr}(c\varphi_\beta : L_\alpha \rightarrow L_\alpha) = \begin{cases} (\text{card } H) \text{Tr}(c\varphi_v : L'_u \rightarrow L'_u), & \text{if } w = w_{\alpha,\beta}, \\ 0, & \text{otherwise,} \end{cases}$$

where u, v are arbitrary elements of $q^{-1}(\alpha), q^{-1}(\beta)$. Similarly, the homomorphism $\varphi_{\alpha^{-1}c} = \varphi_{u^{-1}c}$ sends each direct summand L'_v of L_β (with $v \in q^{-1}(\beta)$) into $L'_{u^{-1}wvu}$. Therefore

$$\begin{aligned} \text{Tr}(\varphi_{\alpha^{-1}c} : L_\beta \rightarrow L_\beta) &= \sum_{v \in q^{-1}(\beta), v = u^{-1}wvu} \text{Tr}(\varphi_{u^{-1}c} : L'_v \rightarrow L'_v) \\ &= \begin{cases} (\text{card } H) \text{Tr}(\varphi_{u^{-1}c} : L'_v \rightarrow L'_v), & \text{if } w = w_{\alpha, \beta}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Now, axiom (3.2.4) for L' implies (3.2.4) for L .

This construction can be combined with those discussed in Section 3. Consider for concreteness the structure of a crossed π' -algebra in the group ring $K[\pi']$ determined by the trivial class $0 \in H^2(\pi', K^*)$. By the argument above, this gives a structure of a crossed π -algebra in the same ring $L = K[\pi']$. Here π acts by conjugations and the inner product is given by $(u, v) \mapsto \delta_{v^{-1}u}^u$ for $u, v \in \pi'$ where δ is the Kronecker delta. The crossed π -algebra L is semisimple if and only if the group ring $L_1 = K[H]$ is semisimple. This is for instance the case if K is an algebraically closed field. If $K = \mathbf{Q}$ and H is non-trivial then $K[H]$ is not semisimple. It is curious to note that the inner product in L admits deformations in the class of crossed π -algebras. Namely, choose a non-degenerate symmetric bilinear form $\mu : K[H] \times K[H] \rightarrow K$ on $L_1 = K[H]$ such that the pair $(K[H], \mu)$ is a Frobenius algebra. (There are many such forms as it is easy to see for cyclic H .) We define the inner product $\eta_\mu : L \otimes L \rightarrow K$ by $\eta_\mu(L_\alpha \otimes L_\beta) = 0$ if $\alpha\beta \neq 1$ and $\eta_\mu(a, b) = \mu(ab, 1_L)$ for $a \in L_\alpha, b \in L_{\alpha^{-1}}$ and any $\alpha \in \pi$. Here $ab \in L_1 = K[H]$ and $1_L \in K[H]$ is the unit element of L . It is easy to check that L with this inner product is a crossed π -algebra.

If K has a ring involution $k \mapsto \bar{k} : K \rightarrow K$ and μ satisfies $\mu(u^{-1}, v^{-1}) = \overline{\mu(u, v)}$ for any $u, v \in H$ then the antilinear involution in $L = K[\pi']$ sending each element of π' to its inverse defines a Hermitian structure on L .

10.4. Crossed group-algebras from algebra automorphisms. Let us consider crossed π -algebras L such that $L_\alpha = 0$ for all $\alpha \neq 1$. Axioms (3.1.1) - (3.2.3) amount to saying that (L_1, η) is a commutative Frobenius algebra with an action of π by algebra automorphisms preserving η . Let us call such a pair (L_1, π) a π -F-algebra. Axiom (3.2.4) means that the action of π is traceless in the sense that $\text{Tr}(c\varphi_\alpha : L_1 \rightarrow L_1) = 0$, for any $\alpha \neq 1, c \in L_1$. Thus any traceless π -F-algebra (L_1, π) gives rise to a crossed π -algebra such that $L_\alpha = 0$ for $\alpha \neq 1$. Note that the tensor product $L_1 \otimes L'_1$ of two π -F-algebras $(L_1, \pi), (L'_1, \pi)$ is a π -F-algebra. If (L_1, π) or (L'_1, π) is traceless then $(L_1 \otimes L'_1, \pi)$ is traceless.

π -F-algebras naturally arise in the study of groups of homeomorphisms. Consider a closed connected oriented even-dimensional manifold M and set $L_1 = L_1(M) = \oplus_{k \text{ even}} H^k(M; \mathbf{Q})$. The product in L_1 is the cup-product and the form η is defined by $\eta(a, b) = (a \cup b)([M])$. Clearly, L_1 is a Frobenius algebra. Now, any group π of orientation preserving self-homeomorphisms of M acts on L_1 via

induced homomorphisms. This action preserves the grading in L_1 and therefore $\text{Tr}(c\alpha_* : L_1 \rightarrow L_1) = 0$ for all $\alpha \in \pi$ and $c \in \oplus_{k \neq 0, k \text{ even}} H^k(M; \mathbf{Q}) \subset L_1$. The only requirement arises for $c = 1 \in H^0(M; \mathbf{Q})$ and consists in the identity $\text{Tr}(\alpha_* : L_1 \rightarrow L_1) = 0$ for all $\alpha \in \pi, \alpha \neq 1$. For instance, consider an orientation-reversing involution of the $2n$ -dimensional sphere $j_n : S^{2n} \rightarrow S^{2n}$. It is clear that the action of j_n on $H^*(S^{2n}; \mathbf{Q})$ is traceless. Therefore for any orientation-reversing involution j of a closed connected oriented even-dimensional manifold M , the product $j_n \times j : S^{2n} \times M \rightarrow S^{2n} \times M$ induces a traceless endomorphism of $L_1(S^{2n} \times M)$. In particular, $j_n \times j_m$ is a traceless endomorphism of $L_1(S^{2n} \times S^{2m})$. This yields examples of traceless $\mathbf{Z}/2\mathbf{Z}$ -F-algebras and hence of crossed $\mathbf{Z}/2\mathbf{Z}$ -algebras.

This example can be extended in a slightly different direction. Consider a closed connected oriented even-dimensional manifold M and a fixed point free group π of orientation preserving self-homeomorphisms of M . By the Lefschetz theorem, the super-trace of the action of π in the graded space $\oplus_k H^k(M; \mathbf{Q})$ is zero. If M has only even-dimensional cohomology, this gives an example of a traceless π -F-algebra. In general this suggests to consider a wider class of crossed super- π -algebras.

Deformations of Frobenius algebras form a subject of a deep theory based on the Witten-Dijkgraaf-Verlinde-Verlinde equation (see for instance [Du]). It would be interesting to generalize this theory to crossed group-algebras.

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